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# Fuzzy differential equations

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**Abstract**

This paper first presents a new solution to the fuzzy first-order initial value problem. Elementary properties of this new solution are given. We then compare various derivatives of fuzzy functions that have been presented in the literature followed by comparing the different solutions one may obtain to the fuzzy initial value problem using these derivatives. Examples are given, including linear and non-linear fuzzy first-order differential equations, where we find both our new solution and other solutions to the fuzzy initial value problem. © 2000 Published by Elsevier Science B.V. All rights reserved.

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## 1. Introduction

In this paper we will consider the first-order ordinary differential equation

$$\mathrm{d}y/\mathrm{d}t = f(t, y, k), \quad y(0) = c, \tag{1}$$

where $k = (k_1, \dots, k_n)$ is a vector of constants, and $t$ is in some interval (closed and bounded) $I$ which contains zero. Extensions to $y, c, f$ all vectors is mentioned in the last section.

We assume that $f$ satisfies conditions [5, p. 223; 22, p. 24] so that Eq. (1) has an unique solution $y = g(t, k, c)$, for $t \in I$, $k \in K \subset R^n$, $c \in C \subset R$. Let $I_1$, be an interval for the $y$-values and set $\mathscr{R} = I \times I_1$, a region in $\mathbb{R}^2$. Well-known sufficient conditions for Eq. (1) to have a unique solution are, given any $k \in K$ and $c \in C$: (1) $(0, c)$ is in $\mathscr{R}$; (2) $f$ is continuous in $\mathscr{R}$ ($k$ is held fixed); and (3) $\partial f/\partial y$ is continuous in $\mathscr{R}$. If these conditions are satisfied, then there is a unique solution $y = g(t, k, c)$ for $t \in I^*$. Since zero will belong to $I^*$ we will assume that $I^* = I$. We will also assume that $g$ is continuous on $I \times K \times C$.

The values of the $k_i$ and $c$ are uncertain and we will model this uncertainty by substituting triangular fuzzy numbers for the $k_i$ and $c$ in Eq. (1). We then wish to solve for $y$ which will now be a fuzzy function. Our new solution for fuzzy $y$ is the topic of this paper. Initial results about our new solution were first presented in [1].

The rest of this section presents the basic notation we will use and then: (1) the second section defines our new solution; (2) the third section compares some of various derivatives of fuzzy functions that have been presented in the literature; (3) the fourth section then compares the various solutions to the fuzzy initial value problem that one may obtain using the different

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derivatives of Section 3; (4) the fifth section contains applications to linear and non-linear fuzzy first-order differential equations; and (5) the final section has a brief summary and our conclusions.

We place a bar over a capital letter to denote a fuzzy subset of $R^n$. So, $\overline{Y}, \overline{K}, \overline{C}$, etc. all represent fuzzy subsets of $R^n$ for some $n$. We write $\overline{A}(x)$, a number in $[0,1]$, for the membership function of $\overline{A}$ evaluated at $x \in R^n$. Define $\overline{A} \leqslant \overline{B}$ when $\overline{A}(x) \leqslant \overline{B}(x)$ for all $x$. An $\alpha$-cut of $\overline{A}$, written $\overline{A}[\alpha]$, is defined as $\{x \mid \overline{A}(x) \geqslant \alpha\}$, for $0<\alpha \leqslant 1$. We separately specify $\overline{A}[0]$ as the closure of the union of all the $\overline{A}[\alpha]$ for $0<\alpha \leqslant 1$.

We adopt the general definition of a fuzzy number given in [10]. A triangular fuzzy number $\overline{N}$ is defined by three numbers $a_1<a_2<a_3$ where the graph of $\overline{N}(x)$ is a triangle with base on the interval $[a_1,a_3]$ and vertex at $x=a_2$. We specify $\overline{N}$ as $(a_1/a_2/a_3)$. We will write: (1) $\overline{N}>0$ if $a_1>0$; (2) $\overline{N} \geqslant 0$ if $a_1 \geqslant 0$; (3) $\overline{N}<0$ if $a_3<0$; and (4) $\overline{N} \leqslant 0$ if $a_3 \leqslant 0$. The $\alpha$-cut of any fuzzy number is always a closed and bounded interval.

Let $\overline{K}=(\overline{K_1},\dots,\overline{K_n})$ be a vector of triangular fuzzy numbers and let $\overline{C}$ be another triangular fuzzy number. Substitute $\overline{K}$ for $k$ and $\overline{C}$ for $c$ in Eq. (1) and we get

$$\mathrm{d}\overline{Y}/\mathrm{d}t = f(t,\overline{Y},\overline{K}), \quad \overline{Y}(0)=\overline{C}, \tag{2}$$

assuming we have adopted some definition for the derivative of the unknown fuzzy function $\overline{Y}(t)$. We wish to solve Eq. (2) for $\overline{Y}(t)$ and have $\overline{Y}(t)$ a fuzzy number for each $t$ in $I$. In general, we use the notation $\mathrm{d}\overline{Y}/\mathrm{d}t$ for the derivative of a fuzzy function $\overline{Y}$, although we have not yet defined this derivative. In Section 3 we will introduce special notation, for various derivatives of fuzzy functions, which have appeared in the literature.

## 2. New solution

Our new solution concept is based on our ideas in solving fuzzy equations [3]. Let $K(\alpha)=\overline{K_1}[\alpha]\times\cdots\times\overline{K}_n[\alpha]$ and $\Phi(\alpha)=K(\alpha)\times\overline{C}[\alpha]$, for $0\leqslant\alpha\leqslant 1$. We assume that $\Phi(0)\subset K\times C$ so that $g$ will be continuous on $I\times\Phi(\alpha)$ for all $\alpha$. We first fuzzify the crisp solution $y=g(t,k,c)$ to obtain $\overline{Y}(t)=g(t,\overline{K},\overline{C})$ using the extension principle. Alternatively, we get $\alpha$-cuts as follows [2, 3]:

$$\overline{Y}(t)[\alpha]=[y_1(t,\alpha),y_2(t,\alpha)], \tag{3}$$

with

$$y_1(t,\alpha)=\min\{g(t,k,c)\mid k\in\overline{K}[\alpha],\ c\in\overline{C}[\alpha]\} \tag{4}$$

and

$$y_2(t,\alpha)=\max\{g(t,k,c)\mid k\in\overline{K}[\alpha],\ c\in\overline{C}[\alpha]\}, \tag{5}$$

for $t\in I$ and $\alpha\in[0,1]$. Still another equivalent procedure to determine $\overline{Y}(t)$ is to first specify, for $0\leqslant\alpha\leqslant 1$, and $t\in I$

$$\Omega(\alpha)=\{g(t,k,c)\mid (k,c)\in\Phi(\alpha)\}, \tag{6}$$

and then define the membership function of $\overline{Y}(t)$ as follows:

$$\overline{Y}(t)(x)=\sup\{\alpha\mid x\in\Omega(\alpha)\}. \tag{7}$$

**Theorem 2.1.** 1. *$\overline{Y}(t)[\alpha]=\Omega(\alpha)$ for all $\alpha\in[0,1]$, $t\in I$.*

2. *$\overline{Y}(t)$ is a fuzzy number for all $t\in I$.*

**Proof.** 1. Proposition 1 follows from Theorem 2 in [2].

2. The proof that $\overline{Y}(t)$ is a fuzzy number is similar to the proof of Theorem 1 in [3] and is omitted.

Assume that $y_i(t,\alpha)$ is differentiable with respect to $t\in I$ for each $\alpha$ in $[0,1]$, $i=1,2$. We write the partial of $y_i(t,\alpha)$ with respect to $t$ as $y_i'(t,\alpha)$, $i=1,2$. Let

$$\Gamma(t,\alpha)=[y_1'(t,\alpha),y_2'(t,\alpha)], \tag{8}$$

for $t\in I$, $\alpha\in[0,1]$. If $\Gamma(\alpha)$ defines the $\alpha$-cuts of a fuzzy number for each $t\in I$ we will say that $\overline{Y}(t)$ is *differentiable* and write

$$\frac{\mathrm{d}\overline{Y}(t)}{\mathrm{d}t}[\alpha]=\Gamma(t,\alpha)=[y_1'(t,\alpha),y_2'(t,\alpha)], \tag{9}$$

for all $t\in I$, $\alpha\in[0,1]$. Notice, that Eq. (9) is just the derivative (with respect to $t$) of Eq. (3). So, Eq. (9) could be written $\mathrm{d}/\mathrm{d}t(\overline{Y}(t)[\alpha])$. Sufficient conditions for $\Gamma(t,\alpha)$ to define the $\alpha$-cuts of a fuzzy number are [10, 11]

- $y_1'(t,\alpha)$ and $y_2'(t,\alpha)$ are continuous on $I\times[0,1]$; [2]
- $y_1'(t,\alpha)$ is an increasing function of $\alpha$ for each $t\in I$;

[2] We will assume later (the continuity condition) that $y_i'(t,\alpha)$ is continuous for $i=1,2$.

- $y_2'(t,\alpha)$ is a decreasing function of $\alpha$ for each $t \in I$; and
- $y_1'(t,1) \leqslant y_2'(t,1)$ all $t \in I$.

Now for $\overline{Y}(t)$ to be a solution to the fuzzy initial value problem we need that $d\overline{Y}(t)/dt$ exists but also Eq. (2) must hold. To check Eq. (2) we must first compute $f(t,\overline{Y},\overline{K})$. $\alpha$-cuts of $f(t,\overline{Y},\overline{K})$ can be found as follows:

$$f(t,\overline{Y},\overline{K})[\alpha] = [f_1(t,\alpha), f_2(t,\alpha)], \tag{10}$$

with

$$f_1(t,\alpha) = \min\{f(t,y,k) \mid y \in \overline{Y}(t)[\alpha],\ k \in \overline{K}[\alpha]\}, \tag{11}$$

$$f_2(t,\alpha) = \max\{f(t,y,k) \mid y \in \overline{Y}(t)[\alpha],\ k \in \overline{K}[\alpha]\}, \tag{12}$$

for $t \in I$, $\alpha \in [0,1]$. We will say that $\overline{Y}$ is a solution to Eq. (2) if $d\overline{Y}(t)/dt$ exists and

$$y_1'(t,\alpha) = f_1(t,\alpha), \tag{13}$$

$$y_2'(t,\alpha) = f_2(t,\alpha), \tag{14}$$

$$y_1'(0,\alpha) = c_1(\alpha), \tag{15}$$

$$y_2'(0,\alpha) = c_2(\alpha), \tag{16}$$

where $\overline{C}[\alpha] = [c_1(\alpha), c_2(\alpha)]$. We show in Section 5 that $\overline{Y}(t)$ does solve certain fuzzy initial value problems. But first, let us now investigate other methods of differentiating fuzzy functions.

## 3. Derivatives

Let $\overline{X}(t)$ = a fuzzy number for each $t \in I$. Also, let $\overline{X}(t)[\alpha] = [x_1(t,\alpha), x_2(t,\alpha)]$ and write $x_i'(t,\alpha)$ for the partial of $x_i(t,\alpha)$ with respect to $t$, $i = 1,2$. We assume these partials always exist in this section. Now we will discuss the Goetchel–Voxman derivative, the Seikkala derivative, the Dubois–Prade derivative, the Puri–Ralescu derivative, and the Kandel–Friedman–Ming derivative of $\overline{X}(t)$. Other authors [4, 16–19, 23] have also discussed the derivative of a fuzzy function, however these approaches (except [23] which is similar to [10]) are more abstract and not directly applicable to solving the fuzzy initial value problem we have in mind in Section 5. Therefore, we will not discuss any further in this paper the results presented in [4, 16–19, 23].

### 3.1. Goetschel–Voxman derivative

The Goetschel–Voxman derivative of $\overline{X}(t)$, written $GVD\overline{X}(t)$, was defined in [10] and discussed in [7, 9, 15]. The definition of $GVD\overline{X}(t)$ is given in Appendix A. We know that [10, Theorem 2.2] if $GVD\overline{X}(t)$ exists, then

$$GVD\overline{X}(t)[\alpha] = [x_1'(t,\alpha), x_2'(t,\alpha)], \tag{17}$$

for all $t \in I$, $\alpha \in [0,1]$. However, $GVD\overline{X}(t)$ may not be a fuzzy number for some $t$ in $I$ and non-standard fuzzy subtraction ($-\overline{N}$ is not a fuzzy number when $\overline{N}$ is a fuzzy number) is used in the definition of the derivative.

### 3.2. Seikkala derivative

The Seikkala derivative of $\overline{X}(t)$, written $SD\overline{X}(t)$, was defined in [21]. This definition was as follows: if $[x_1'(t,\alpha), x_2'(t,\alpha)]$ are the $\alpha$-cuts of a fuzzy number for each $t \in I$, then $SD\overline{X}(t)$ exists and $SD\overline{X}(t)[\alpha] = [x_1'(t,\alpha), x_2'(t,\alpha)]$.

Notice that this is the definition of the derivative of a fuzzy function we used in the previous section. That is, if $d\overline{Y}(t)/dt$ exists, then $SD\overline{X}(t) = d\overline{Y}(t)/dt$. Also, $SD\overline{X}(t)$ is a fuzzy number for all $t \in I$.

### 3.3. Dubois–Prade derivative

The Dubois–Prade derivative of $\overline{X}(t)$, written $DPD\overline{X}(t)$, was defined and discussed in [6, 7]. $DPD\overline{X}(t)$ always exists and its membership function is given by

$$DPD\overline{X}(t)(x) = \sup\{\alpha \mid x = x_1'(t,\alpha),\ x = x_2'(t,\alpha)\}. \tag{18}$$

However, $DPD\overline{X}(t)$ may not be a fuzzy number.

Let us consider the situation where $DPD\overline{X}(t)$ can be a fuzzy number for $t$ in $I$. Assume that $x_1'(t,\alpha)$ and $x_2'(t,\alpha)$ satisfy the sufficient conditions for

$[x_1'(t,\alpha), x_2'(t,\alpha)]$ to define $\alpha$-cuts of a fuzzy number given in Section 2. We may have to add something to the definition of $DPD\overline{X}(t)$ to obtain a fuzzy number. If $x_1'(t,1) < x_2'(t,1)$ for some value of $t$, then we separately define $DPD\overline{X}(t)(x) = 1$ for all $x$ satisfying $x_1'(t,1) < x < x_2'(t,1)$. The $DPD\overline{X}(t)$ will be a fuzzy number.

### 3.4. Puri–Ralescu derivative

The Puri–Ralescu derivative of $\overline{X}(t)$, written $PRD\overline{X}(t)$, was defined in [20] and studied in [11–13]. The definition of $PRD\overline{X}(t)$ is presented in Appendix A. We know [20, Proposition 3.1] that if $PRD\overline{X}(t)$ exists, then

$$PRD\overline{X}(t)[\alpha] = [x_1'(t,\alpha), x_2'(t,\alpha)], \quad (19)$$

for all $t \in I$, all $\alpha \in [0,1]$. $PRD\overline{X}(t)$ is always a fuzzy number for each $t \in I$. However, non-standard fuzzy subtraction is used in that they employ the Hukuhara difference of fuzzy sets.

### 3.5. Kandel–Friedman–Ming derivative

The Kandel–Friedman–Ming derivative of $\overline{X}(t)$, written $KFMD\overline{X}(t)$ was defined in [8, 14]. We give the definition of $KFMD\overline{X}(t)$ in Appendix A. We also have [14, Corollary 1] that when this derivative exists

$$KFMD\overline{X}(t)[\alpha] = [x_1'(t,\alpha), x_2'(t,\alpha)], \quad (20)$$

for all $t \in I$, all $\alpha \in [0,1]$. This derivative also equals a fuzzy number for all $t \in I$. Also, non-standard fuzzy subtraction is used in this paper in their definition of the derivative.

### 3.6. Relationships

In this subsection we investigate the relationships between these five derivatives.

**Theorem 3.1.** 1. *If $GVD\overline{X}(t)$ exists and is a fuzzy number for each $t \in I$, then $SD\overline{X}(t)$ exists and $GVD\overline{X}(t) = SD\overline{X}(t)$.*

2. *If $PRD\overline{X}(t)$ exists, then $SD\overline{X}(t)$ exists and $PRD\overline{X}(t) = SD\overline{X}(t)$.*

3. *If $KFMD\overline{X}(t)$ exists, then so does $SD\overline{X}(t)$ and they are equal.*

4. *If $SD\overline{X}(t)$ exists and if $x_1'(t,\alpha)$ and $x_2'(t,\alpha)$ are both continuous in $\alpha$ for each $t$ in $I$, then $SD\overline{X}(t) = DPD\overline{X}(t)$.*

**Proof.** 1. This follows from Eq. (17) and the definition of $SD\overline{X}(t)$.

2. This follows from Eq. (19) and the definition of $SD\overline{X}(t)$.

3. This follows from Eq. (20) and the definition of $SD\overline{X}(t)$.

4. If $SD\overline{X}(t)$ exists, $SD\overline{X}(t)$ is a fuzzy number and therefore: (1) $x_1'(t,\alpha)$ is an increasing function of $\alpha$ for each $t \in I$; (2) $x_2'(t,\alpha)$ is a decreasing function of $\alpha$ for each $t$ in $I$; and (3) $x_1'(t,1) < x_2'(t,1)$ for all $t$.

We need to show that for each $t$, $DPD\overline{X}(t)(x) = SD\overline{X}(t)(x)$. Consider the cases: (1) $x < x_1'(t,0)$; (2) $x_1'(t,0) \leqslant x \leqslant x_1'(t,1)$; (3) $x_1'(t,1) < x < x_2'(t,1)$; (4) $x_2'(t,1) \leqslant x \leqslant x_2'(t,0)$ and (5) $x_2'(t,0) < x$. In case (1) and (5) both memberships are zero. In case (3) both are one. In case (2) and (4) there is a unique value of $\alpha$, say $\alpha^*$, so that $x_1'(t,\alpha^*) = x$ ($x_2'(t,\alpha^*) = x$). Then both equal $\alpha^*$ because of the continuity assumption.

We will need the following definition in the remainder of the paper. We will say the *continuity condition* holds when $x_i'(t,\alpha)$ is continuous on $I \times [0,1]$, $i = 1,2$. The continuity condition will hold in all the applications we are interested in within Section 5. In the following theorem we only assume the existence of $SD\overline{X}(t)$.

**Theorem 3.2.** *Assume the continuity condition holds. If $SD\overline{X}(t)$ exists, then $SD\overline{X}(t) = DPD\overline{X}(t) = GVD\overline{X}(t) = PRD\overline{X}(t) = KFMD\overline{X}(t)$.*

**Proof.** Assume $SD\overline{X}(t)$ exists.

1. Show $SD\overline{X}(t) = GVD\overline{X}(t)$. By Theorem 2.3 in [10] we have $GVD\overline{X}(t)$ exists and then Eq. (17) holds. But $SD\overline{X}(t)$ exists so that Eq. (17) defines the $\alpha$-cuts of a fuzzy number. Hence, $GVD\overline{X}(t)$ is a fuzzy number and Theorem 3.1 says that they are equal. ($SD\overline{X}(t) = GVD\overline{X}(t)$).
2. $SD\overline{X}(t) = DPD\overline{X}(t)$ from Theorem 3.1.
3. We first show that $KFMD\overline{X}(t)$ exists, then by Theorem 3.1 they are equal. Let

$$H_i(t,h,\alpha) = \frac{x_i(t+h,\alpha) - x_i(t,\alpha)}{h} - x_i'(t,\alpha), \quad (21)$$

for $i = 1,2$, $\alpha \in [0,1]$, $t \in I$ and $t+h \in I$. From [14] we need to show

$$\lim_{h\to 0}\left(\int_0^1 |H_i(t,h,\alpha)|^p \,\mathrm{d}\alpha\right)^{1/p} = 0,$$

$i = 1,2$ and $t \in I$. Then $\alpha$-cuts of $KFMD\overline{X}(t)$ are given by $[x_1'(t,\alpha), x_2'(t,\alpha)] = SD\overline{X}(t)[\alpha]$. We show the limit for $i = 1$ only. Now $x_1'(t,\alpha)$ is uniformly continuous on $I \times [0,1]$ ($I = [0,M]$ for an arbitrary large number $M$), so given $\varepsilon > 0$ there is a $\delta > 0$ so that $0 \leqslant |\zeta| < \delta$ implies

$$|x_1'(t+\zeta,\alpha) - x_1'(t,\alpha)| < \varepsilon$$

for all $\alpha \in [0,1]$, $t \in I$ (assume $t+\zeta \in I$ also). By the mean value theorem if $0 < |h| < \delta$

$$|H_1(t,h,\alpha)| = |x_1'(t^*,\alpha) - x_1'(t,\alpha)| < \varepsilon,$$

where $t^*$ lies between $t$ and $t+h$ ($|t^*-t| < \delta$). So $(\int_0^1 |H_1(t,h,\alpha)|^p \,\mathrm{d}\alpha)^{1/p} < \varepsilon$ if $0 < |h| < \delta$. Hence, the limit is zero for $i = 1$.

4. Show $SD\overline{X}(t) = PRD\overline{X}(t)$. We show $PRD\overline{X}(t)$ exists, then by Theorem 3.1 they are equal. We will use Theorem 5.1 in [11]. There are two thing to verify. The first is:
   4.1. For $0 \leqslant h < \beta$, some $\beta > 0$

$$z_1(t,h,\alpha) = x_1(t+h,\alpha) - x_1(t,\alpha), \tag{22}$$

$$z_2(t,h,\alpha) = x_2(t+h,\alpha) - x_2(t,\alpha) \tag{23}$$

are the $\alpha$-cuts of a fuzzy number, and

$$w_1(t,h,\alpha) = x_1(t,\alpha) - x_1(t-h,\alpha), \tag{24}$$

$$w_2(t,h,\alpha) = x_2(t,\alpha) - x_2(t-h,\alpha) \tag{25}$$

also are the $\alpha$-cuts of a fuzzy number. Here $t \in I = [0,\infty)$, $t+h$ and $t-h \in I$, $\alpha \in [0,1]$. We will show the result for only $z_i$, $i = 1,2$. By the mean value theorem

$$\frac{z_1(t,h,\alpha)}{h} = x_1'(t_1^*,\alpha) \tag{26}$$

for $t_1^* \in (t,t+h)$ and

$$\frac{z_2(t,h,\alpha)}{h} = x_2'(t_2^*,\alpha) \tag{27}$$

for $t_2^* \in (t,t+h)$.

But $x_1'(t,\alpha)$ is increasing in $\alpha$ and $h \geqslant 0$ implies that $z_1(t,h,\alpha)$ is increasing in $\alpha$. Similar, $z_2(t,h,\alpha)$ is decreasing in $\alpha$. Also $z_1(t,h,\alpha)$ and $z_2(t,h,\alpha)$ are continuous. Finally, $z_1(t,h,1) = hx_1'(t,1) \leqslant hx_2'(t,1) = z_2(t,h,1)$.

   4.2. Let $I = [0,M]$, $M > 0$. We need to show, for $t \in I$, $\varepsilon > 0$, there is a $\delta > 0$, so that

$$|H_i(t,h,\alpha)| < \varepsilon, \tag{28}$$

$i = 1,2$ and if

$$L(t,h,\alpha) = \frac{x_i(t,\alpha) - x_i(t-h,\alpha)}{h} - x_i'(t,\alpha), \tag{29}$$

$i = 1,2$, then also

$$|L(t,h,\alpha)| < \varepsilon, \tag{30}$$

$i = 1,2$, for all $\alpha \in [0,1]$, if $0 \leqslant h < \delta$. But this follows from part 3 above. This completes the proof of Theorem 3.2.

**Theorem 3.3.** *Assume the continuity condition holds. If one of the derivatives SD, GVD and it is a fuzzy number, PRD, or KFMD exist, then so do the others and they are all equal.*

**Proof.** Follows from Theorems 3.1 and 3.2.

## 4. Fuzzy initial value problem

In this section we will look at solutions to the fuzzy initial value problem (FIVP) (see Eq. (2)) using $SD\overline{X}(t)$, $RPD\overline{X}(t)$ and $KFMD\overline{X}(t)$. We will not consider $GVD\overline{X}(t)$ and $DPD\overline{X}(t)$ since these derivatives are not necessarily equal to a fuzzy number.

### *4.1. Buckley–Feuring solution*

The Buckley–Feuring solution, written BFS, to the FIVP, was defined in the second section. To review those results let $BFS = \overline{Y}(t)$. Then: (1) $\overline{Y}(t) = g(t,\overline{K},\overline{C})$ (Eqs. (3)–(5)); (2) $SD\overline{Y}(t)$ exists (Eq. (8) defines a fuzzy number for all $t$); and (3) $SD\overline{Y}(t) = f(t,\overline{Y}(t),\overline{K})$ and $\overline{Y}(0) = \overline{C}$ (Eqs. (13)–(16)). We have the following results regarding $BFS = \overline{Y}(t)$.

**Theorem 4.1.** *Assume* $SD\overline{Y}(t)$ *exists for* $t \in I$. *Then* $BFS = \overline{Y}(t)$ *if*

$$\frac{\partial f}{\partial y} > 0, \qquad \frac{\partial g}{\partial c} > 0, \tag{31}$$

*and*

$$\left(\frac{\partial g}{\partial k_i}\right)\left(\frac{\partial f}{\partial k_i}\right) > 0, \tag{32}$$

$i = 1, \ldots, n$. *If Eq.* (31) *does not hold or Eq.* (32) *does not hold for some i, then* $\overline{Y}(t)$ *does not solve the FIVP.*

**Proof.** Let us assume there is only one $k_i = k$ and that $\partial g/\partial k < 0$ and $\partial f/\partial k < 0$. The proof for $\partial g/\partial k > 0$ and $\partial f/\partial k > 0$ is similar and omitted. Since $\partial g/\partial k < 0$ and $\partial g/\partial c > 0$ we have $y_1(t,\alpha) = g(t, k_2(\alpha), c_1(\alpha))$, $y_2(t,\alpha) = g(t, k_1(\alpha), c_2(\alpha))$. Also, because $\partial f/\partial y > 0$ and $\partial f/\partial k < 0$ we see that $f_1(t,\alpha) = f(t, y_1(t,\alpha), k_2(\alpha))$ and $f_2(t,\alpha) = f(t, y_2(t,\alpha), k_1(\alpha))$. Now, $y = g(t,k,c)$ is the unique solution to $\mathrm{d}y/\mathrm{d}t = f(t,y,k)$ and $y(0) = c$ which implies that

$$g'(t,k,c) = f(t, g(t,k,c), k), \tag{33}$$

and

$$g(0,k,c) = c. \tag{34}$$

Assuming $SD\overline{Y}(t)$ exists we see that

$$\begin{aligned} y_1'(t,\alpha) &= g'(t, k_2(\alpha), c_1(\alpha)) \\ &= f(t, g(t, k_2(\alpha), c_1(\alpha)), k_2(\alpha)) \\ &= f_1(t,\alpha), \end{aligned}$$

and

$$\begin{aligned} y_1(0,\alpha) &= g(0, k_2(\alpha), c_1(\alpha)) \\ &= c_1(\alpha), \end{aligned}$$

and also

$$\begin{aligned} y_2'(t,\alpha) &= g'(t, k_1(\alpha), c_2(\alpha)) \\ &= f(t, g(t, k_1(\alpha), c_2(\alpha)), k_1(\alpha)) \\ &= f_2(t,\alpha), \end{aligned}$$

and

$$\begin{aligned} y_1(0,\alpha) &= g(0, k_1(\alpha), c_2(\alpha)) \\ &= c_2(\alpha), \end{aligned}$$

for all $\alpha \in [0,1]$ and $t \in I$. Hence Eqs. (13)–(16) hold.

Now consider the situation where Eq. (31) or (32) does not hold. Let us only look at one case where $\partial f/\partial y < 0$ (assume $\partial g/\partial c > 0$, $\partial f/\partial k > 0$, $\partial g/\partial k > 0$). Then we have $f_1(t,\alpha) = f(t, y_2(t,\alpha), k_1(\alpha))$, $f_2(t,\alpha) = f(t, y_1(t,\alpha), k_2(\alpha))$, $y_1(t,\alpha) = g(t, k_1(\alpha), c_1(\alpha))$, and $y_2(t,\alpha) = g(t, k_2(\alpha), c_2(\alpha))$. Eq. (13) becomes $y_1'(t,\alpha) = g'(t, k_1(\alpha), c_1(\alpha)) = f_1(t,\alpha) = f(t, g(t, k_2(\alpha), c_2(\alpha)), k_1(\alpha))$ which is not true.

### 4.2. *Seikkala solution*

The Seikkala solution, written SS, to the FIVP is $\overline{X}(t)$ if $SD\overline{X}(t)$ exists and

$$SD\overline{X}(t) = f(t, \overline{X}(t), \overline{K}), \tag{35}$$

$$\overline{X}(0) = \overline{C}. \tag{36}$$

Going back to the definition of $SD\overline{X}(t)$, Section 3.2, we see that Eqs. (35) and (36) are equivalent to Eqs. (13)–(16) substituting $x_i$ for $y_i$. That is,

$$x_1'(t,\alpha) = f_1(t,\alpha), \tag{37}$$

$$x_2'(t,\alpha) = f_2(t,\alpha), \tag{38}$$

$$x_1(0,\alpha) = c_1(\alpha), \tag{39}$$

$$x_2(0,\alpha) = c_2(\alpha), \tag{40}$$

where the $f_i$ are defined as in Eqs. (11) and (12) using $\overline{X}(t)$ for $\overline{Y}(t)$.

### 4.3. *Puri–Ralescu solution*

The Puri–Ralescu solution to the FIVP is $\overline{X}(t)$, written $PRS = \overline{X}(t)$, if $PRD\overline{X}(t)$ exists and $PRD\overline{X}(t) = f(t, \overline{X}(t), \overline{K})$ and $\overline{X}(0) = \overline{C}$. From the results on $PRD\overline{X}(t)$ from Section 3 we see that the equations $PRD\overline{X}(t) = f(t, \overline{X}(t), \overline{K})$ and $\overline{X}(0) = \overline{C}$ are also equivalent to Eqs. (37)–(40).

### 4.4. Kandel–Friedman–Ming solution

The Kandel–Friedman–Ming solution to the FIVP is $\overline{X}(t)$, written $KFMS=\overline{X}(t)$, if $KFMD\overline{X}(t)$ exists and $KFMD\overline{X}(t)=f(t,\overline{X}(t),\overline{K})$ and $\overline{X}(0)=\overline{C}$. From Section 3.5 we see that these equations are also equivalent to Eqs. (37)–(40).

### 4.5. Relationships

We are now interested in determining the relationships between $BFS, SS, PRS$ and $KFMS$.

**Theorem 4.2.** 1. *If $BFS=\overline{Y}(t)$, then $SS=\overline{Y}(t)$.*
2. *If $PRS=\overline{X}(t)$, then $SS=\overline{X}(t)$.*
3. *If $KFMS=\overline{X}(t)$, then $SS=\overline{X}(t)$.*

**Proof.** (1) Follows from the definition of $BFS$ and $SS$.
(2) Follows from Theorem 3.1 and the definition of $PRS$ and $SS$.
(3) Follows from Theorem 3.1 and the definition of $KFMS$ and $SS$.

**Theorem 4.3.** 1. *If $BFS=\overline{Y}(t)$ and $PRD\overline{Y}(t)$ exists, then $PRS=\overline{Y}(t)$.*
2. *If $BFS=\overline{Y}(t)$ and $KFMD\overline{Y}(t)$ exists, then $KFMS=\overline{Y}(t)$.*

**Proof.** 1. If $BFS=\overline{Y}(t)$, then $SS=\overline{Y}(t)$ from Theorem 4.2. If $PRD\overline{Y}(t)$ exists, then $PRS$ is a solution to Eqs. (37)–(40). But $SS$ is also a solution to Eqs. (37)–(40). Hence $PRS=SS=\overline{Y}(t)$.
2. Similar to the proof of 1. above and is omitted.

**Theorem 4.4.** *Assume the continuity condition holds and $I=[0,M]$, some $M>0$.*
1. *If $SD\overline{X}(t)$ exists, then $SS=PRS=KFMS$.*
2. *If $BFS=\overline{Y}(t)$ and the continuity condition holds for $\overline{Y}(t)$, then $BFS=SS=PRS=KFMS$.*

**Proof.** 1. Assume the $x_i'(t,\alpha)$ is continuous, $i=1,2$. Then by Theorem 3.2, $SD\overline{X}(t)=PRD\overline{X}(t)=KFMD\overline{X}(t)$. Each solution must solve Eqs. (37)–(40). Hence all solutions are the same.
2. Assume $y_i'(t,\alpha)$ are continuous, $i=1,2$. If $BFS=\overline{Y}(t)$, then by Theorem 4.2, $SS=\overline{Y}(t)$ and by part 1, all solutions are equal.

We will say that the *derivative condition* holds for the initial value problem when Eqs. (31) and (32) are true.

**Theorem 4.5.** 1. *If $PRS=\overline{X}(t)$ and the derivative condition holds, then $BFS=\overline{X}(t)$.*
2. *If $SS=\overline{X}(t)$ and the derivative condition holds, then $BFS=\overline{X}(t)$.*
3. *If $KFMS=\overline{X}(t)$ and the derivative condition holds, then $BFS=\overline{X}(t)$.*

**Proof.** All proofs are similar, so we will only show part 1. If $PRS=\overline{X}(t)$ then $PRD\overline{X}(t)$ exists and Eqs. (37)–(40) hold. Assume one $k_i=k$ and $\partial f/\partial k>0$, $\partial g/\partial k>0$ (the other cases are similar and are omitted).

We see

$$y_1(t,\alpha)=g(t,k_1(\alpha),c_1(\alpha)),$$

$$y_2(t,\alpha)=g(t,k_2(\alpha),c_2(\alpha)),$$

$$f_1(t,\alpha)=f(t,y_1(t,\alpha),k_1(\alpha)),$$

$$f_2(t,\alpha)=f(t,y_2(t,\alpha),k_2(\alpha)).$$

Also, $y=g(t,k,c)$ is the unique solution to $\mathrm{d}y/\mathrm{d}t=f(t,y,k)$, $y(0)=c$. Now look at Eqs. (37)–(40):

$$x_1'(t,\alpha)=f(t,x_1(t,\alpha),k_1(\alpha)),$$

$$x_1(0,\alpha)=c_1(\alpha),$$

implies that $x_1(t,\alpha)=g(t,k_1(\alpha),c_1(\alpha))=y_1(t,\alpha)$ and

$$x_2'(t,\alpha)=f(t,x_2(t,\alpha),k_2(\alpha)),$$

$$x_2(0,\alpha)=c_2(\alpha),$$

implies that $x_2(t,\alpha)=g(t,k_2(\alpha),\ c_2(\alpha))=y_2(t,\alpha)$. Therefore $BFS=\overline{X}(t)$.

### 4.6. Discussion

Theorem 4.2 implies that the Seikkala solution is the most general solution to the fuzzy initial value problem. Theorems 4.3 and 4.5 relate our new solution to the other solutions. Theorem 4.4 (since the continuity condition holds) will have an important application in the next section because it indicates that we should first search for the *BFS* and then, assuming that the *BFS* solution does not exist, look for the *SS*.

## 5. Applications

Throughout this section the continuity condition will hold and we will assume that $I=[0,M]$, for some $M>0$, so Theorem 4.4 will dictate our strategy for solving the FIVP. The strategy is: (1) find $\overline{Y}(t)$ and check to see if $BFS=\overline{Y}(t)$ because if this is true all solutions equal $\overline{Y}(t)$; (2) if $SD\overline{Y}(t)$ does not exist or $\overline{Y}(t)$ does not solve the FIVP, find *SS* (if it exists) because if $SS=\overline{X}(t)$, then all the other solutions (*PRS*, *KFMS*) will be equal to $\overline{X}(t)$.

### 5.1. Growth/decay model

Consider the initial value problem $\mathrm{d}y/\mathrm{d}t=f(t,y,k)=ky$ with $y(0)=c$. We know that $y=g(t,k,c)=c\exp(kt)$. Checking the derivative condition (Theorem 4.1) we have $\overline{Y}(t)$ (assuming $SD\overline{Y}(t)$ exists) solve the FIVP if $\partial f/\partial y=k>0$ since then $\partial g/\partial k>0$, $\partial g/\partial c>0$ and $\partial f/\partial k=y>0$ for all $t$. So, if $k<0$, we look for a *SS*.

**Example 1.** Assume $k>0$. Let $\overline{K}=(k_1/k_2/k_3)$, a triangular fuzzy number with $k_1>0$, and $\overline{C}=(c_1/c_2/c_3)$ also with $c_1\geqslant 0$ ($c$ is usually positive in a growth problem). Also set $\overline{K}[\alpha]=[k_1(\alpha),k_2(\alpha)]$, $\overline{C}[\alpha]=[c_1(\alpha),c_2(\alpha)]$. Then we easily see that $\alpha$-cuts of $\overline{Y}(t)$ are

$$y_1(t,\alpha)=c_1(\alpha)\exp(k_1(\alpha)t), \tag{41}$$

$$y_2(t,\alpha)=c_2(\alpha)\exp(k_2(\alpha)t). \tag{42}$$

Therefore, Eq. (8) becomes

$$\Gamma(t,\alpha)=[c_1(\alpha)k_1(\alpha)\exp(k_1(\alpha)t),\; c_2(\alpha)k_2(\alpha)\exp(k_2(\alpha)t)], \tag{43}$$

which is easily seen to be the $\alpha$-cuts of a fuzzy number $SD\overline{Y}(t)$ for all $t\in I$. Next we see that $\overline{Y}(t)$ solves the FIVP. We compute

$$f_1(t,\alpha)=c_1(\alpha)k_1(\alpha)\exp(k_1(\alpha)t), \tag{44}$$

$$f_2(t,\alpha)=c_2(\alpha)k_2(\alpha)\exp(k_2(\alpha)t), \tag{45}$$

so that Eqs. (13) and (14) hold. Obviously, $\overline{Y}(0)=\overline{C}$ and therefore $BFS=\overline{Y}(t)$.

We notice that $y_i(t,0)\to\infty$, for $i=1,2$, $(y_2(t,0)-y_1(t,0))\to\infty$, and $y_1(t,1)=y_2(t,1)\to\infty$ as $t\to\infty$. Hence, the support of $\overline{Y}(t)$ grows larger and larger, and the place where $\overline{Y}(t)(x)=1$ gets larger and larger, as $t$ grows.

**Example 2.** Assume $k<0$. $\overline{K}$ and $\overline{C}$ are triangular fuzzy numbers as in Example 1 with $k_3<0$. We need to solve the following system (Eqs. (37)–(40)) using $k_i(\alpha)<0$:

$$x_1'(t,\alpha)=k_1(\alpha)x_2(t,\alpha), \tag{46}$$

$$x_2'(t,\alpha)=k_2(\alpha)x_1(t,\alpha), \tag{47}$$

$$x_1(0,\alpha)=c_1(\alpha), \tag{48}$$

$$x_2(0,\alpha)=c_2(\alpha) \tag{49}$$

for $x_1(t,\alpha)$ and $x_2(t,\alpha)$. In Eqs. (46) and (47) we have assumed a positive solution for $\overline{X}(t)$, or $x_1(t,\alpha)\geqslant 0$. If the intervals $[x_1(t,\alpha),x_2(t,\alpha)]$ define $\alpha$-cuts of a fuzzy number $\overline{X}(t)$, then $SS=\overline{X}(t)$. The system may, or may not, have a Seikkala solution.

The general solution to Eqs. (46)–(49) is

$$x_1(t,\alpha)=A_{11}(\alpha)\exp(\omega t)+A_{12}(\alpha)\exp(-\omega t), \tag{50}$$

$$x_2(t,\alpha)=A_{21}(\alpha)\exp(\omega t)-A_{22}(\alpha)\exp(-\omega t), \tag{51}$$

where $\omega=\sqrt{k_1(\alpha)k_2(\alpha)}$, $\lambda=\sqrt{k_1(\alpha)/k_2(\alpha)}$, $A_{11}(\alpha)=0.5(c_1(\alpha)+\lambda c_2(\alpha))$, $A_{12}(\alpha)=0.5(c_1(\alpha)-\lambda c_2(\alpha))$, $A_{21}(\alpha)=0.5(c_1(\alpha)/\lambda+c_2(\alpha))$ and $A_{22}(\alpha)=0.5(c_1(\alpha)/\lambda-c_2(\alpha))$. We need to check if $[x_1(t,\alpha),x_2(t,\alpha)]$ defines the $\alpha$-cuts of a fuzzy number $\overline{X}(t)$ for all $t\in I$. But this is not true since for large $t$ $x_1(t,\alpha)\approx A_{11}(\alpha)\exp(\omega t)$, $x_2(t,\alpha)\approx A_{21}(\alpha)\exp(\omega t)$ and $A_{11}(\alpha)>A_{21}(\alpha)$ for all $\alpha$. So, in general *SS* does not exist for large values of $t$. *SS* exists for $0\leqslant t\leqslant\gamma$ for some $\gamma>0$.

### 5.2. General linear first order

Here we consider $\mathrm{d}y/\mathrm{d}t=-k_1y+k_2$, $y(0)=c$ so that $y=g(t,k,c)=k_2/k_1+(c-k_2/k_1)\exp(-k_1t)$. Now $f(t,y,k_1,k_2)=-k_1y+k_2$. It can make a difference whether we look at $\mathrm{d}y/\mathrm{d}t+k_1y=k_2$ or $\mathrm{d}y/\mathrm{d}t=-k_1y+k_2$ but first we will concentrate on the form $\mathrm{d}y/\mathrm{d}t=f(t,y,k_1,k_2)=-k_1y+k_2$. The derivative condition (Theorem 4.1) says we need $\partial f/\partial k > 0$, or $k_1 < 0$ for a BFS. So assume that $k_1 < 0$. Next, we easily see that $\partial f/\partial k_2 > 0$, $\partial g/\partial c > 0$, and $\partial g/\partial k_2 > 0$ (since $k_1 < 0$). Also, $\partial f/\partial k_1 = -y$ is negative since $y > 0$ and if $k_2 > 0$ we find that $\partial g/\partial k_1 < 0$. So we assume that $k_1 < 0$, $k_2 > 0$ and *BFS* will exist as long as $SD\overline{Y}(t)$ exists. If $k_1 > 0$, we would look for Seikkala solution.

**Example 3.** Let $\overline{K_1} = (k_{11}/k_{12}/k_{13})$ with $k_{13} < 0$, $\overline{K_2} = (k_{21}/k_{22}/k_{23})$ and $k_{21} > 0$, and $\overline{C} = (c_1/c_2/c_3)$, $c_1 > 0$. Also set $\overline{K_i}[\alpha] = [k_{i1}(\alpha), k_{i2}(\alpha)]$, $i = 1, 2$. Then α-cuts of $\overline{Y}(t)$ are

$$y_1(t,\alpha) = \frac{k_{21}(\alpha)}{k_{12}(\alpha)} + \left(c_1(\alpha) + \frac{k_{21}(\alpha)}{k_{12}(\alpha)}\right)\exp(-k_{12}(\alpha)t), \tag{52}$$

$$y_2(t,\alpha) = \frac{k_{22}(\alpha)}{k_{11}(\alpha)} + \left(c_2(\alpha) + \frac{k_{22}(\alpha)}{k_{11}(\alpha)}\right)\exp(-k_{11}(\alpha)t), \tag{53}$$

since $\partial g/\partial k_1 < 0$, $\partial g/\partial k_2 > 0$ and $\partial g/\partial c > 0$. Now, we may see that

$$y_1'(t,\alpha) = (k_{21}(\alpha) - k_{12}(\alpha)c_1(\alpha))\exp(-k_{12}(\alpha)t), \tag{54}$$

$$y_2'(t,\alpha) = (k_{22}(\alpha) - k_{11}(\alpha)c_2(\alpha))\exp(-k_{11}(\alpha)t), \tag{55}$$

defines α-cuts of a fuzzy number $SD\overline{Y}(t)$ because $k_{21}'(\alpha) > 0$, $k_{12}'(\alpha) < 0$, $c_1'(\alpha) > 0$, $k_{22}'(\alpha) < 0$, $k_{11}'(\alpha) > 0$ and $c_2'(\alpha) < 0$ where the prime denotes the derivative on α. Hence it follows that $\overline{Y}(t) = BFS$.

We again see that $y_i(t,0) \to \infty$, $(y_2(t,0) - y_1(t,0)) \to \infty$, and $y_1(t,1) = y_2(t,1) \to \infty$ as $t \to \infty$. So, as $t$ grows both the support of $\overline{Y}(t)$ and the position where $\overline{Y}(t)(x) = 1$ get larger and larger.

**Example 4.** Now assume that $k_1 > 0$ and let $\overline{K_1} = (k_{11}/k_{12}/k_{13})$ with $k_{11} > 0$. We need to solve $(\overline{K_2} > 0)$ $(\overline{C} \geqslant 0)$

$$x_1'(t,\alpha) = -k_{12}(\alpha)x_2(t,\alpha) + k_{21}(\alpha), \tag{56}$$

$$x_2'(t,\alpha) = -k_{11}(\alpha)x_1(t,\alpha) + k_{12}(\alpha), \tag{57}$$

$$x_1'(0,\alpha) = c_1(\alpha), \tag{58}$$

$$x_2'(0,\alpha) = c_2(\alpha), \tag{59}$$

and see if $[x_1(t,\alpha), x_2(t,\alpha)]$ defines α-cuts of fuzzy number $\overline{X}(t)$.

The solution is

$$x_1(t,\alpha) = A_{11}(\alpha)\exp(\omega t) + A_{12}(\alpha)\exp(-\omega t) + \lambda_1, \tag{60}$$

$$x_2(t,\alpha) = A_{21}(\alpha)\exp(\omega t) + A_{22}(\alpha)\exp(-\omega t) + \lambda_2, \tag{61}$$

where $\omega = \sqrt{k_{11}(\alpha)k_{12}(\alpha)}$, $\lambda_1 = k_{22}(\alpha)/k_{11}(\alpha)$, $\lambda_2 = k_{21}(\alpha)/k_{12}(\alpha)$ and $A_{11}(\alpha) = [(c_1 - \lambda_1)\mu - (c_2 - \lambda_2)]/2\mu$, $A_{12}(\alpha) = [(c_2 - \lambda_2) + \mu(c_1 - \lambda_1)]/2\mu$, $A_{21}(\alpha) = -\mu A_{11}(\alpha)$, $A_{22}(\alpha) = -\mu A_{12}(\alpha)$ where $\mu = \sqrt{k_{11}(\alpha)/k_{12}(\alpha)}$.

Do Eqs. (60) and (61) define the α-cuts of a fuzzy number for $t$ in $I$? We would like to show for each $t$ in $I$ that: (1) $x_1(t,\alpha)$ is an increasing function of α; (2) $x_2(t,\alpha)$ is a decreasing function of α; and (3) $x_1(t,1) \leqslant x_2(t,1)$. But we have been unable to do this. What we have been able to do is to randomly generate two million values of $\overline{K}_1 > 0$, $\overline{K}_2 > 0$, and $\overline{C} \geqslant 0$ and then check to see if Eqs. (60) and (61) do define the α-cuts of a fuzzy number for values of $t$ in $I$. In all cases the answer was yes. So, it looks like the Seikkala solution exists in this case.

Assume that Eqs. (60) and (61) do define a fuzzy number which we will call $\overline{X}(t)$. We will now investigate what happens to $\overline{X}(t)$ as $t$ grows larger and larger. So, we will assume $M$ is a very large positive number so that $t \in I = [0, M]$ can become a large positive number. Let $c_1(1) = c_2(1) = c$, $k_{11}(1) = k_{12}(1) = k_1$ and $k_{21}(1) = k_{22}(1) = k_2$. At $\alpha = 1$ we have $\lambda_1 = \lambda_2 = k_2/k_1 = \lambda$, $\mu = 1$, and $\omega = k_1$. We see that $x_1(t,1) = x_2(t,1) = \lambda + (c - \lambda)\exp(-k_1 t)$ so as $t$ grows we get $x_1(t,1) = x_2(t,1) \approx \lambda$. Next, we argue that $x_1(t,\alpha) \to -\infty$, $x_2(t,\alpha) \to \infty$ as $t \to \infty$, for $\alpha < 1$. First, we see that $x_1(t,\alpha) \approx A_{11}(\alpha)\exp(\omega t) + \lambda_1$, and $x_2(t,\alpha) \approx A_{21}(\alpha)\exp(\omega t) + \lambda_2$ as $t$ gets larger and larger. It is easy to check that $A_{11}(\alpha) < 0$ and $A_{21}(\alpha) > 0$ for $\alpha < 1$. Since $\exp(\omega t) \to \infty$ as $t \to \infty$ the results follows. This implies that $\overline{X}(t) \approx \overline{1}$ for large $t$ where $\overline{1}(x) = 1$ for all $x$. That is, the uncertainty in the solution grows in time until it is completely uncertain.

**Example 5.** This example shows that if we change the initial value problem slightly, the *BFS* can exist for $k_1 > 0$. Consider the following mixing problem. The tank initially contains 300 gals of brine which has dissolved in it $c$ lbs of salt. Coming into the tank at 3 gals/min is brine with concentration

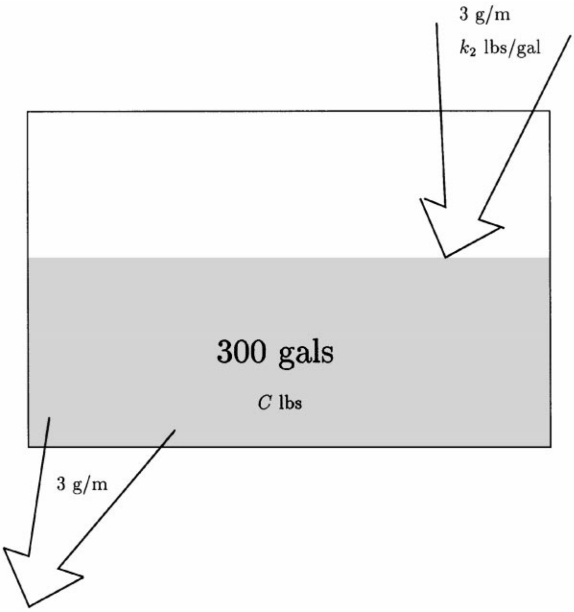


Fig. 1. Mixing problem.

$k_2$ lbs salt/gals and the well stirred mixture leaves at the rate 3 gals/min. See Fig. 1. Let $y(t)=$ lbs of salt in the tank at any time $t \geqslant 0$. Then

$$\frac{dy}{dt} + \frac{1}{100}y = 3k_2, \quad y(0) = c. \tag{62}$$

The solution is

$$y = g(t, k_2, c) = 300k_2 + (c - 300k_2)\exp\left(-\frac{1}{100}t\right). \tag{63}$$

Notice that we wrote the initial value problem differently, we did not write $dy/dt = f(t, y, k_2)$ because then there will be no BFS.

We next fuzzify $g$ to get $\overline{Y}(t)$. Assume $\overline{K}_2 > 0$ and $\overline{C} > 0$. Since $\partial g/\partial k_2 > 0$, $\partial g/\partial c > 0$ we obtain the $\alpha$-cuts of $\overline{Y}(t)$

$$y_1(t,\alpha) = 300k_{21}(\alpha) + (c_1(\alpha) - 300k_{21}(\alpha))\exp(-0.01t), \tag{64}$$

$$y_2(t,\alpha) = 300k_{22}(\alpha) + (c_2(\alpha) - 300k_{22}(\alpha))\exp(-0.01t). \tag{65}$$

Then we compute $SD\overline{Y}(t)$, whose $\alpha$-cuts are

$$y_1'(t,\alpha) = (3k_{21}(\alpha) - 0.01c_1(\alpha))\exp(-0.01t), \tag{66}$$

$$y_2'(t,\alpha) = (3k_{22}(\alpha) - 0.01c_2(\alpha))\exp(-0.01t), \tag{67}$$

which defines a fuzzy number if $3k_{21}' - 0.01c_1' > 0$, $3k_{22}' - 0.01c_2' < 0$. Let $\overline{K_2} = (k_{21}/k_{22}/k_{23})$, $\overline{C} = (c_1/c_2/c_3)$. Then $k_{21}' = k_{22} - k_{21}$, $c_1' = c_2 - c_1$, $k_{22}' = k_{22} - k_{23}$ and $c_2' = c_2 - c_3$. Let us then assume that $3(k_{22} - k_{21}) > 0.01(c_2 - c_1)$ and $3(k_{23} - k_{22}) > 0.01(c_3 - c_2)$ so that $SD\overline{Y}(t)$ exists. Now we check to see if $SD\overline{Y}(t) + 0.01\overline{Y}(t) = 3\overline{K}_2$. Obviously, this is true so that $BFS = \overline{Y}(t)$. Notice that $\overline{Y}(t) \to 300\overline{K}_2$ as $t \to \infty$.

Notice that $SD\overline{Y}(t)$ does not equal $-0.01\overline{Y}(t) + 3\overline{K}_2(t)$. Hence solutions to the fuzzy initial value problem are sensitive to how it is written down. If you change how the fuzzy initial value problem is presented it may, or may not, have a *BFS* or a *SS*. This of course is a result of fuzzy arithmetic in that $-\overline{N}$ reverses a fuzzy number $\overline{N}$.

### 5.3. First-order non-linear

Consider the initial value problem

$$\frac{dy}{dt} = k_1y^2 + k_2, \quad y(0) = 0, \tag{68}$$

where $k_i > 0$ for $i = 1, 2$. The solution is

$$y = g(t, k_1, k_2, c) = \lambda\tan(\omega t), \tag{69}$$

on $I = [0, \lambda)$ with $\lambda = \pi/(2\omega)$ where $\omega = \sqrt{k_1k_2}$ and $\lambda = \sqrt{k_2/k_1}$. Now consider the corresponding fuzzy initial value problem with $\overline{K}_i > 0$ for $i = 1, 2$. We calculate $\overline{Y}(t)$ using $\partial g/\partial k_1 > 0$ and $\partial g/\partial k_2 > 0$

$$y_1(t,\alpha) = \lambda_1(\alpha)\tan(\omega_1(\alpha)t), \tag{70}$$

$$y_2(t,\alpha) = \lambda_2(\alpha)\tan(\omega_2(\alpha)t), \tag{71}$$

with $\lambda_1(\alpha) = \sqrt{k_{21}(\alpha)/k_{11}(\alpha)}$, $\lambda_2(\alpha) = \sqrt{k_{22}(\alpha)/k_{12}(\alpha)}$, $\omega_1(\alpha) = \sqrt{k_{11}(\alpha)k_{21}(\alpha)}$ and $\omega_2(\alpha) = \sqrt{k_{12}(\alpha)k_{22}(\alpha)}$. Then $\alpha$-cuts of $SD\overline{X}(t)$ are

$$y_1'(t,\alpha) = k_{21}(\alpha)\sec^2(\omega_1(\alpha)t), \tag{72}$$

$$y_2'(t,\alpha) = k_{22}(\alpha)\sec^2(\omega_2(\alpha)t), \tag{73}$$

which defines a fuzzy number. Then one may easily check that $SD\overline{X}(t) = \overline{K}_1\overline{Y}^2(t) + \overline{K}_2$, so *BFS* exists for this non-linear fuzzy initial value problem.

Actually, once we knew that $SD\overline{X}(t)$ existed we could have used Theorem 4.1 to conclude $BFS$ existed because $\partial f/\partial y > 0$ and $(\partial f/\partial k_i)(\partial g/\partial k_i) > 0$ for $i = 1, 2$. If $k_1 < 0$, then $BFS$ does not exist for this problem.

## 6. Summary and conclusions

This paper is concerned with methods of solving the fuzzy initial value problem given in Eq. (2). We first developed our new solution $\overline{Y}(t)$. Basically, $\overline{Y}(t)$ is the fuzzification of the crisp solution to the initial value problem in Eq. (1). We gave necessary and sufficient conditions for $\overline{Y}(t)$ to solve the fuzzy initial value problem.

We also investigated various methods, which have been presented in the literature, of differentiating fuzzy functions and we developed relationships between these derivatives. Using these methods of differentiating fuzzy functions, after discarding those which do not always produce fuzzy numbers for the derivative, we investigated the types of solutions they can produce to the fuzzy initial value problem. We then presented relationships between these solutions and our new solution $\overline{Y}(t)$.

Examples are given including a non-linear fuzzy initial value problem and the general linear fuzzy initial value problem. Assuming a certain continuity condition holds (give in Section 3.6) the solution strategy was: (1) first see if our new solution solves the fuzzy initial value problem because if it does, then all solutions will equal our new solution; (2) if our new solution does not solve the problem, find the Seikkala solution (defined in Section 4.2) because if it exists, then all the other solutions will equal the Seikkala solution. That is, under the continuity, there are only two types of solutions. We also showed that if our new solution does not exist, then you might be able to rewrite the differential equation in another form, and then our new solution will exist.

Extensions to other fuzzy problems is too difficult. For example we can apply our new solution concept to fuzzy partial differential equations. We first find the crisp solution, fuzzify it and then check to see if it satisfies the fuzzy partial differential equation. This will be a topic for future research.

## Appendix A

In this Appendix we present the definitions of $GVD\overline{X}(t)$, $PRD\overline{X}(t)$ and $KFMD\overline{X}(t)$.

### A.1. Goetschel–Voxman derivative

We first must give the metric used for this derivative. Let $\overline{X}(t)$ and $\overline{Z}(t)$ be two fuzzy functions for $t \in I$. Both $\overline{X}(t)$ and $\overline{Z}(t)$ are fuzzy numbers for each $t$ in $I$. Set $\overline{X}(t)[\alpha] = [x_1(t,\alpha), x_2(t,\alpha)]$ and $\overline{Z}(t)[\alpha] = [z_1(t,\alpha), z_2(t,\alpha)]$ for all $t$ and $\alpha$. Then the metric $D$ is

$$D(\overline{X}(t), \overline{Z}(t)) = \sup_{\alpha}\{\max[|x_1(t,\alpha) - z_1(t,\alpha)|, |x_2(t,\alpha) - z_2(t,\alpha)|]\}, \quad (A.1)$$

for all $t$ in $I$.

The derivative of $\overline{X}(t)$ at $t_0$ defined as

$$GVD\overline{X}(t_0) = \lim_{h\to 0}\left(\frac{\overline{X}(t_0+h) - \overline{X}(t_0)}{h}\right), \quad (A.2)$$

provided the limit exists with respect to the metric $D$. However, the subtraction in Eq. (A.2) is not standard fuzzy subtraction because

$$[\overline{X}(t_0+h) - \overline{X}(t_0)][\alpha] = [x_1(t_0+h,\alpha) - x_1(t_0,\alpha), x_2(t_0+h,\alpha) - x_2(t_0,\alpha)], \quad (A.3)$$

for all $t$, $\alpha$. Standard fuzzy arithmetic would produce

$$[x_1(t_0+h,\alpha) - x_2(t_0,\alpha), x_2(t_0+h,\alpha) - x_1(t_0,\alpha)]. \quad (A.4)$$

### A.2. Puri–Ralescu derivative

Again, we first specify the metric used for this derivative. $\overline{X}(t)$ and $\overline{Z}(t)$ are as in the previous section and the metric $D$ is now

$$D(\overline{X}(t), \overline{Z}(t)) = \sup_{\alpha} H(\overline{X}(t)[\alpha], \overline{Z}(t)[\alpha]), \quad (A.5)$$

for all $t$, where $H$ is the Hausdorff metric on non-empty compact subsets of $\mathbb{R}$.

Next, we need to define the Hukuhara difference between two fuzzy numbers $\overline{A}$ and $\overline{B}$. If there exists a fuzzy number $\overline{C}$ so that $\overline{C}+\overline{A}=\overline{B}$, then $\overline{C}$ is called the Hukuhara difference between $\overline{B}$ and $\overline{A}$ and we write this as

$$\overline{B} \overset{*}{-} \overline{A} = \overline{C}. \tag{A.6}$$

$\overline{X}(t)$ is differentiable at $t_0$ in $I$ if there exists fuzzy number $PRD\overline{X}(t_0)$ so that

$$\lim_{h\to 0^+} \left( \frac{\overline{X}(t_0+h) \overset{*}{-} \overline{X}(t_0)}{h} \right) = PRD\overline{X}(t_0), \tag{A.7}$$

and

$$\lim_{h\to 0^+} \left( \frac{\overline{X}(t_0) \overset{*}{-} \overline{X}(t_0-h)}{h} \right) = PRD\overline{X}(t_0). \tag{A.8}$$

Both limits are taken with respect to the metric $D$ in Eq. (A.5).

### A.3. Kandel–Friedman–Ming derivative

First, fuzzy numbers now do not need to have compact support. The metric $D$ used is

$$D_p(\overline{X}(t), \overline{Z}(t)) = \max\left\{ \left[ \int_0^1 |x_1(t,\alpha) - z_1(t,\alpha)|^p \, d\alpha \right]^{1/p}, \left[ \int_0^1 |x_2(t,\alpha) - z_2(t,\alpha)|^p \, d\alpha \right]^{1/p} \right\}, \tag{A.9}$$

for $x_1(t,\alpha)$, $x_2(t,\alpha)$, $z_1(t,\alpha)$ and $z_2(t,\alpha)$ all in $L_p[0,1]$, for all $t$ in $I$.

$\overline{X}(t)$ is differentiable at $t_0 \in I$ if there is a fuzzy number $KFMD\overline{X}(t_0)$ so that

$$\lim_{h\to 0} D_p \left[ \frac{\overline{X}(t_0+h) - \overline{X}(t_0)}{h}, KFMD\overline{X}(t_0) \right] = 0. \tag{A.10}$$

However, the subtraction $\overline{X}(t_0+h) - \overline{X}(t_0)$ in the above equation is not standard fuzzy subtraction since it is defined as in Eq. (A.3).

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