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Invariance and Stability of Fuzzy Systems

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1. INTRODUCTION

The elaboration of a stability theory of fuzzy systems is fully justified by the use, in a variety of practical situations, of fuzzy controllers [11, 27, 28]. However, such a theory may be founded upon ideas that arise from a historical study of dynamical systems theory.

During the last quarter of the nineteenth century, the study of ordinary differential equations underwent some rather radical changes. Prior to this period, the major emphasis in the subject had been on the methods of solving equations. It was during this period that Peano [20] gave a rigorous proof of the existence theorem for solutions of ordinary differential equations. During the same period, Lipschitz [15] and Picard [22] showed that the method of successive approximations led to a proof of the existence and the uniqueness of solutions to the initial value problem. These developments represent the *dénouement* of the attempts to solve differential equations.

As the works of Peano, Lipschitz and Picard were finishing one chapter in the book of differential equations, another was being initiated with the research of Liapunov [14] and Poincaré [23]. This new chapter was based on an entirely different approach, where one did not attempt to compute the solutions; rather one tried to exploit their topological properties.

Poincaré, in his famous *mémoire* [23], was the first to look at an ordinary differential equation from the point of view of the geometry of the trajectories, thereby creating the topological theory of ordinary differential equations. Almost at the same time, the stability theory created by Liapunov [14] although heavily quantitative in its methods, stressed the importance of some topological features of ordinary differential equations. During the 50 years following Liapunov and Poincaré, many important advances were made. It is with the works of Birkhoff [4], Nemytskii and Stepanov [19], and Smale [26] that researchers realized that the essence of this theory was based on the notion of dynamical system.

Based on Liapunov's original works, "classical" stability theory is concerned with the equilibrium points of the system and the dynamical

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behavior of this system in a neighborhood of these points. However, a feature that characterises these stability notions is that they pertain to one specific model of the system under consideration. Important as such an approach is in engineering, its utility in systems arising in biology, economics and social sciences must be viewed with some scepticism. The basic problem is that systems of this kind almost always operate far from equilibrium and are subject to perturbations that change the equilibrium points. Moreover, even if a system has a fixed structure, due to the uncertainty attached with its mathematical model, it is impossible to determine its structure with exactitude.

Based upon Poincaré's pioneering works, the "modern" stability counterpart to the equilibrium-centered classical view is the concept of a structurally stable system [16, 21], i.e., of a system whose behavior is not drastically altered by a slight change in its structure.

These two aspects of the stability problem are of equal importance. They play complementary roles in the analysis of a dynamical system. Thus, it is desirable to combine them, i.e., to perform an analysis of the behavior of a family of trajectories generated by all models near to the nominal model. Since, in addition, some models represent the system with greater fidelity than others, this problem is clearly related to the concept of fuzzy system.

Based on our theory of fuzzy systems introduced in [8], this paper may be viewed as an attempt to solve this problem.

Having defined, in Section 2, the fuzzy derivative of a real-valued function, we discuss, in Section 3, the problem of invariance for a fuzzy system. To start with, we define the concept of limit set $\Omega(u)$ of a fuzzy system. Then, we establish the topological properties of $\Omega(u)$. Finally, we propose an Invariance Principle for fuzzy systems, which generalizes the celebrated LaSalle's Invariance Principle [13]. In Section 4, we investigate the problems relative to the stability of a fuzzy system. On the one hand, the concept of α-stability is introduced, removing the binary distinction between stability and instability. On the other hand, in order to allow the trajectories to have different behaviors, two types of α-stability are considered. Based on the inspection of the sign of the fuzzy derivative of a real-valued function V, criteria of α-stability are derived both for V continuous and V lower-semicontinuous. These results, while integrating some aspects of Poincaré's ideas, generalizes what is known as Liapunov stability theory [13, 14].

Throughout this paper, the following notations have been used.

A *fuzzy subset* [31] A of a carrier set X is defined by its *membership function* $\mu_A: X \to I = [0, 1] \subset \mathbb{R}$. The family of all fuzzy subsets of X is denoted by $\mathbf{P}(X)$.

The *α-cut* of a fuzzy subset A of X is, for $\alpha \in I - \{0\}$, the set

$$A_\alpha = \{x \in X: \mu_A(x) \geqslant \alpha\}.$$

Let X be a Banach space. The dynamical behavior of a continuous-time *fuzzy system*, whose state at time t is denoted by $x(t) \in \mathbf{P}(X)$, is governed by the differential equation

$$\mu_{\dot{x}(t)}(\dot{u}) = \bigvee_{u} [\mu_{x(t)} \wedge \mu_R(u, \dot{u})], \qquad t \in \mathbb{R}_+,$$

where R is a fuzzy relation over X (i.e., a fuzzy subset of X^2).

Defining the mapping

$$u \mapsto R_\alpha(u) = \{w \in X: (u, w) \in R_\alpha\}$$

and assuming, if d denotes the metric in X, that, for all $\alpha \in I - \{0\}$,

(i) $R_\alpha(u)$ is compact and nonempty,

(ii) $R_\alpha(\cdot)$ is Lipschitz-continuous, i.e., that there exists a real k such that, for all u_1, u_2, $d[R_\alpha(u_1), R_\alpha(u_2)] < k \cdot d(u_1, u_2)$, we can show that a fuzzy system is governed by the equation

$$\dot{x}_\alpha(t) = R_\alpha(x_\alpha(t)) \triangleq \bigcup_{u \in x_\alpha(t)} R_\alpha(u)$$

which possesses a unique solution. In other words, if x^0 denotes the initial state, there exists, for all α, a mapping $f_\alpha: P(X) \to P(X)$ such that

$$f_\alpha(x_\alpha^0, t) = x_\alpha(t)$$

and a mapping $f: \mathbf{P}(X) \to \mathbf{P}(X)$ such that

$$f(x^0, t) = x(t).$$

This equation is the *evolution equation* of a fuzzy system.

Henceforth, for the sake of simplicity, we will denote the initial state by x and the state at time t by $f(x, t)$.

We can show that

(i) $f_\alpha(f_\alpha(u, t_1), t_2) = f_\alpha(u, t_1 + t_2)$.

(ii) $(u, t) \mapsto f_\alpha(u, t)$ is continuous.

(iii) $f_\alpha(u, t)$ is compact, for all $u \in X$ and all $t \in \mathbb{R}_+$.

A mapping $\phi_\alpha(u, \cdot): t \mapsto \phi_\alpha(u, t)$ such that

(i) $\phi_\alpha(u, t_2) \in f_\alpha[\phi_\alpha(u, t_1), t_2 - t_1]$,

(ii) $\phi_\alpha(u, \cdot)$ is Lipschitz-continuous,

is called an *α-trajectory* through u. The family of α-trajectories is denoted by Φ_α.

An α-trajectory generates a tangent vector field on X and is a solution to the differential relation

$$\dot{\phi}_\alpha(u,t) \in R_\alpha[\phi_\alpha(u,t)] \qquad \text{a.e.} \quad t \in \mathbb{R}_+.$$

The theory of fuzzy systems, developed in [8] and sketched above, generalizes, in addition to the theory of dynamical systems [2, 3, 9], systems subject to changes in their structure and/or in the initial conditions [23], general control systems [3, 24], systems with nonunique solutions [2], systems defined over a nonassociative time space [1].

For all subset M of X and all $\varepsilon > 0$, we define

$$B(M,\varepsilon) = \{x \in X : d(x,M) < \varepsilon\},$$

$$B[M,\varepsilon] = \{x \in X : d(x,M) \leqslant \varepsilon\},$$

$$S(M,\varepsilon) = \{x \in X : d(x,M) = \varepsilon\}.$$

Cl denotes the closure, while ∂ denotes the boundary.

2. Fuzzy Derivative of a Real-Valued Function

2.1. The most important tool to investigate the question of invariance and stability for a fuzzy system is provided by the determination and the study of a real-valued function and the inspection of the sign of its fuzzy derivative.

Let $V: X \to \mathbb{R}$ be a real-valued function of the state-space X. First assume that V is continuous. Given a fuzzy system defined by its flow f, consider the fuzzy subset $DV(u)$ of X, defined, for all $u \in X$, by

$$DV(u) = \lim_{t \to 0_+} t^{-1}[V(f(u,t)) - V(u)],$$

where the operations in $\mathbf{P}(X)$ are made by using the extension principle [32]. The fuzzy set $DV(u)$ is called the *fuzzy derivative* of V along the fuzzy system f.

Since V is continuous and $f_\alpha(u,t)$ is compact (and consequently $\mu_{f(u,t)}$ is upper-semicontinuous), the α-cut $DV_\alpha(u)$ of $DV(u)$ may be written

$$D_\alpha V(u) = \lim_{t \to 0_+} t^{-1}[V(f_\alpha(u,t)) - V(u)].$$

Since, in addition, for all $z \in f_\alpha(u,t)$, there exists an α-trajectory ϕ_α such that $\phi_\alpha(u,t) = z$, $D_\alpha V(u)$ is defined by

$$D_\alpha V(u) = \left\{ \lim_{t \to 0_+} \frac{V(\phi_\alpha(u,t)) - V(u)}{t} : \phi_\alpha \in \Phi_\alpha \right\},$$

i.e.,

$$D_\alpha V(u) = \left\{ \frac{d^+ V(\phi_\alpha(u, t))}{dt} \bigg|_{t=0} : \phi_\alpha \in \Phi_\alpha \right\}.$$

Thus, $D_\alpha V(u)$ is the family of right-hand-side derivatives of V along the α-trajectories through u, at the origin.

2.2. We now wish to extend this definition to the case where $V: X \to \mathbb{R}$ fails to be continuous and is only lower-semicontinuous.

Although it is not immediately obvious, there are some significant advantages to the dropping of the continuity assumption. One can, for example, choose V as a function of the metric d.

Consider the fuzzy subset $DV(u)$ of X defined by

$$DV(u) = \liminf_{t \to 0_+} t^{-1} [V(f(u, t)) - V(u)].$$

Proceeding as above, we can show that

$$D_\alpha V(u) = \left\{ \liminf_{t \to 0_+} \frac{V(\phi_\alpha(u, t)) - V(u)}{t} : \phi_\alpha \in \Phi_\alpha \right\},$$

i.e., that $D_\alpha V(u)$ is the set of lower right-hand-side Dini derivatives of V along the α-trajectories through u, at the origin.

We note that if V happens to be continuous, the two definitions of $DV(u)$ coincide.

For both V continuous and V lower-semicontinuous, the (lower) right-hand-side derivative of V along an α-trajectory ϕ_α through u will be denoted by $\dot{V}(\phi_\alpha, u)$ so that

$$D_\alpha V(u) = \{ \dot{V}(\phi_\alpha, u) : \phi_\alpha \in \Phi_\alpha \}.$$

3. Invariance Principle

3.1. A problem of prime importance is the study of the asymptotic behavior of a fuzzy system, i.e., of the existence and the location of a fuzzy region S of the state-space that $f(u, t)$ approaches as $t \to \infty$. There may even exist a smallest and nonempty fuzzy subset S that also has this property and, if so, one would like to locate it. In this sense, locating such a fuzzy set S is the best asymptotic information one can hope to obtain. In discussing questions of this type, the concept of limit set of a fuzzy system, which generalizes that of limit set of a system due to Birkhoff [4], is quite useful.

DEFINITION 3.1. The *limit set* $\Omega(u)$ of a fuzzy system is the fuzzy set defined by the property: $A \subset \Omega(u)$ if there exist a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, and a sequence $\{A_n\}$, $\mathbf{P}(X) \ni A_n \subset f(u, t_n)$, such that $\mu_{A_n} \to \mu_A$ as $n \to \infty$. The α-cut $\Omega_\alpha(u)$ of $\Omega(u)$ is called the *α-limit set.*

The α-limit set $\Omega_\alpha(u)$ of a fuzzy system is then defined by: $z \in \Omega_\alpha(u)$ if there exist a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, and a sequence $\{z_n\}$, $z_n \in f_\alpha(u, t)$, such that $z_n \to z$ as $n \to \infty$.

Moreover, since, for all $z_n \in f_\alpha(u, t_n)$, there exists an α-trajectory ϕ_α^n through u such that $\phi_\alpha^n(u, t_n) = z_n$, the definition of the α-limit set can be rephrased: $z \in \Omega_\alpha(u)$ if there exist a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, and a sequence $\{\phi_\alpha^n(u, t_n)\}$ such that $\phi_\alpha^n(u, t_n) \to z$. However, this reformulation does not provide a description of the α-limit set of a fuzzy system in terms of the asymptotic behavior of the α-trajectories.

THEOREM 3.1. *The α-limit set $\Omega_\alpha(u)$ of a fuzzy system is given by the property: $z \in \Omega_\alpha(u)$ if there exist a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, and an α-trajectory ϕ_α through u such that $\phi_\alpha(u, t_n) \to z$ as $n \to \infty$.*

Proof. Let $z \in \Omega_\alpha(u)$ and let $\{t_n\}$ and $\{z_n\}$ be the corresponding sequences. Then, for all $\varepsilon > 0$, there exists N such that, for all $n > N$, $d(z_n, z) < \varepsilon/2$. Hence, for all $n > N$,

$$d(z_n, z_{n+1}) < d(z_n, z) + d(z_{n+1}, z) = \varepsilon.$$

Therefore, there exists an α-trajectory ϕ_α^1 through z_N such that for all $n > N$, $\phi_\alpha^1(z_N, t_n - t_N) = z_n$. Thus, there exist an α-trajectory ϕ_α through u and a sequence $\{t_m\}$, $t_m \to \infty$ as $m \to \infty$, such that $\phi_\alpha(u, t_m) \to z$. ∎

We can give this algebraic definition of the α-limit set an equivalent geometric version.

THEOREM 3.2. *Let $\Omega(u)$ be the limit set of a fuzzy system. Then*

$$\Omega_\alpha(u) = \bigcap_{\tau \in \mathbb{R}_+} \mathrm{Cl} \bigcup_{t \geqslant \tau} f_\alpha(u, t).$$

Proof. Denote by Ω_α^1 and Ω_α^2 the α-limit set given by the definition and the theorem, respectively. Several cases have to be considered.

(i) Suppose that $\Omega_\alpha^2(u) = \varnothing$ and $\Omega_\alpha^1(u) \neq \varnothing$. Then, for all $z \in \Omega_\alpha^1(u)$,

$$z \notin \bigcap_{\tau \in \mathbb{R}_+} \mathrm{Cl} \bigcup_{t \geqslant \tau} f_\alpha(u, t).$$

Hence, there exists $\mathscr{T}$ such that

$$z \notin \mathrm{Cl} \bigcup_{t \geqslant \mathscr{T}} f_\alpha(u, t),$$

i.e., such that

$$z \notin \bigcup_{t \geqslant \mathscr{T}} f_\alpha(u, t).$$

Therefore, for any sequence $\{t_n\}$ such that $t_n > \mathscr{T}$ and $t_n \to \infty$ as $n \to \infty$, one cannot find a sequence $\{z_n\}$ such that $z_n \in f_\alpha(u, t_n)$ and $z_n \to z$. Hence, $\Omega_\alpha^2(u) \neq \varnothing$, which is a contradiction. Thus, $\Omega_\alpha^2(u) = \varnothing$ implies $\Omega_\alpha^1(u) = \varnothing$.

(ii) Suppose that $\Omega_\alpha^1(u) = \varnothing$ and $\Omega_\alpha^2(u) \neq \varnothing$. Then, there exists $z \in X$ such that

$$z \in \bigcap_{\tau \in \mathbb{R}_+} \mathrm{Cl} \bigcup_{t \geqslant \tau} f_\alpha(u, t).$$

Hence, for all integers k,

$$z \in \mathrm{Cl} \bigcup_{t \geqslant k} f_\alpha(u, t),$$

i.e., there exist a sequence $\{t_n\}$ such that $t_n \geqslant k$ and a sequence $\{z_n\}$ such that $z_n \in f_\alpha(u, t_n)$ and $z_n \to z$. If $k \to \infty$, $t_n \to \infty$ and then $z \in \Omega_\alpha^1(u)$. This a contradiction. Thus, $\Omega_\alpha^1(u) = \varnothing$ implies $\Omega_\alpha^2(u) = \varnothing$.

(iii) One can assume that $\Omega_\alpha^1(u)$ and $\Omega_\alpha^2(u)$ are nonempty. Consider $z \in \Omega_\alpha^1(u)$. Then, there exist $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, and $\{z_n\}$, $z_n \in f_\alpha(u, t_n)$, such that $z_n \to z$. Suppose that $z \in \Omega_\alpha^2(u)$, i.e., that

$$z \notin \bigcap_{\tau \in \mathbb{R}_+} \mathrm{Cl} \bigcup_{t \geqslant \tau} f_\alpha(u, t).$$

Then, there exists $\mathscr{T}$ such that

$$z \notin \bigcup_{t \geqslant \mathscr{T}} f_\alpha(u, t).$$

Proceeding as in (i), one can show that the supposition leads to a contradiction. Thus, $z \in \Omega_\alpha^2(u)$ and $\Omega_\alpha^1(u) \subset \Omega_\alpha^2(u)$.

Conversely, consider $z \in \Omega_\alpha^2(u)$. Then, for all integers k, we have

$$z \in \mathrm{Cl} \bigcup_{t \geqslant k} f_\alpha(u, t).$$

Therefore, proceeding as in (ii), one can show that $z \in \Omega_\alpha^1(u)$. Thus, $\Omega_\alpha^2(u) \subset \Omega_\alpha^1(u)$. ∎

Combining Theorems 3.1 and 3.2 yields the geometric definition of $\Omega_\alpha(u)$ in terms of the α-trajectories:

$$\Omega_\alpha(u) = \bigcap_{\tau \in \mathbb{R}_+} \operatorname{Cl} \bigcup_{t \geqslant \tau} \bigcup_{\phi_\alpha \in \Phi_\alpha} \varphi_\alpha(u, t).$$

THEOREM 3.3. *The α-limit set $\Omega_\alpha(u)$ of a fuzzy system is closed.*

Proof. Consider a sequence $\{z_m\}$, $z_m \in \Omega_\alpha(u)$, such that $d(z_m, z) < 1/m$ for some $z \in X$. To each z_m, there corresponds a sequence $\{t_{m,n}\}$ such that $t_{m,n} \to \infty$ as $n \to \infty$ and a sequence $\{z_{m,n}\}$ such that $z_{m,n} \in f_\alpha(u, t_{m,n})$ and $d(z_{m,n}, z_m) < 1/n$. Therefore,

$$d(z_{m,n}, z) < d(z_{m,n}, z_n) + d(z_n, z) < 2/n \to 0 \qquad \text{as} \quad n \to \infty.$$

Thus, $z \in \Omega_\alpha(u)$. ∎

3.2. As alluded to in previous sections, one of the main aspects in the study of the asymptotic behavior of a fuzzy system is the question of invariance.

DEFINITION 3.2. A subset M of X is said to be *α-invariant* for a fuzzy system f if, for all $u \in M$ and for all $t \in \mathbb{R}_+$, $f_\alpha(u, t) \cap M \neq \emptyset$.

Remark 3.1. (i) If M_1 and M_2 are α-invariant, so is $M_1 \cup M_2$.

(ii) The fact that M is α-invariant does not guarantee that, for all $u \in M$, $f_\alpha(u, t)$ does not leave M for all $t \in \mathbb{R}_+$. However, M is α-invariant if and only if, for all $u \in X$, there exists an α-trajectory ϕ_α such that $\phi_\alpha(u, t) \in M$, for all $t \in \mathbb{R}_+$.

THEOREM 3.4. *The α-limit set $\Omega_\alpha(u)$ of a fuzzy system is α-invariant.*

Proof. Let $z \in \Omega_\alpha(u)$ and let $\{t_n\}$ and $\{z_n\}$ be the sequences such that $t_n \to \infty$ as $n \to \infty$, $z_n \in f_\alpha(u, t_n)$ and $z_n \to z$. For all n, there exists an α-trajectory ϕ_α^n such that $z_n = \phi_\alpha^n(u, t_n)$. Let ϕ_α be the mapping defined by

$$\phi_\alpha(\phi_\alpha^n(u, t_n), \cdot): t \mapsto \phi_\alpha^n(u, t_n + t).$$

Then, ϕ_α is an α-trajectory through $\phi_\alpha^n(u, t_n)$. However,

$$\phi_\alpha(\phi_\alpha(u, t_n), 0) = \phi_\alpha(u, t_n) \to z \qquad \text{as} \quad n \to \infty.$$

Therefore, for all $\tau \in \mathbb{R}_+$,

$$\phi_\alpha(\phi_\alpha^n(u, t_n), \tau) = \phi_\alpha^n(u, t_n + \tau) \to \phi_\alpha(z, \tau) \qquad \text{as} \quad n \to \infty.$$

Hence, there exist a sequence $\{t_n'\}$, $t_n' = t_n + \tau \to \infty$ as $n \to \infty$, and a sequence

$\{\phi_\alpha^n(u, t_n')\}$ such that $\phi_\alpha^n(u, t_n') \to \phi_\alpha(z, \tau)$. Thus, we have $\phi_\alpha(z, \tau) \in \Omega_\alpha(u)$ and $f_\alpha(z, \tau) \cap \Omega_\alpha(u) \neq \varnothing$, for all $\tau \in \mathbb{R}_+$. ∎

Define the mapping $\delta: P(X) \times P(X) \to \mathbb{R}_+$ by

$$\delta(A, B) = \sup\{d(u, B): u \in B\}.$$

We note that δ is not a metric since $\delta(A, B) = 0$ if and only if $A \subset B$. However,

$$d(A, B) = \max[\delta(A, B), \delta(B, A)].$$

We note that if $M \subset X$ is such that $\delta[f_\alpha(u, t), M] \to 0$ as $t \to \infty$, then $\Omega_\alpha(u) \subset \mathrm{Cl}\, M$. Therefore, if $\Omega_\alpha(u)$ is not empty and $\delta[f_\alpha(u, t), \Omega_\alpha(u)] \to 0$ as $t \to \infty$, $\Omega_\alpha(u)$ is the smallest set having this property; and one would like to locate it. Unfortunately, it is not necessarily true that $\delta[f_\alpha(u, t), \Omega_\alpha(u)] \to 0$ as $t \to \infty$. In fact, $\Omega_\alpha(u)$ may be empty. This desirable property holds under one additional assumption.

DEFINITION 3.3. Given a fuzzy system f, the fuzzy set

$$\gamma(u) = \bigcup_{t \in \mathbb{R}_+} f(u, t)$$

is called the *hull*. Its α-cut

$$\gamma_\alpha(u) = \bigcup_{t \in \mathbb{R}_+} f_\alpha(u, t)$$

is called the *α-hull*.

THEOREM 3.5. *Let $\gamma(u)$ and $\Omega(u)$ be the hull and the limit set of a fuzzy system. If there exists a lower-semicontinuous function $V: X \to \mathbb{R}_+$ such that*

(i) $V(z) \to \infty$ *as* $\|z\| \to \infty$,

(ii) $\sup D_\alpha V(z) \leqslant 0$, *for all* $z \in X - \Omega_\alpha(u)$,

then, $\gamma_\alpha(u)$ *is bounded.*

Proof. Suppose that $\gamma_\alpha(u)$ is not bounded. Then there exist an α-trajectory ϕ_α and a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\|\phi_\alpha(u, t_n)\| \to \infty$ as $n \to \infty$. Hence, if $u_n = \phi_\alpha(u, t_n)$, we have $V(u_n) \to \infty$ as $n \to \infty$. Therefore, $\dot{V}(\phi_\alpha, u_n) > 0$ and $\sup D_\alpha V(u) > 0$, which is a contradiction. ∎

THEOREM 3.6. *If the α-hull $\gamma_\alpha(u)$ of a fuzzy system f is bounded, then the α-limit set $\Omega_\alpha(u)$ is nonempty and compact. Moreover,*

$\delta[f_\alpha(u,t), \Omega_\alpha(u)] \to 0$ as $t \to \infty$ and $\Omega_\alpha(u)$ is the smallest set having this property.

Proof. (i) As X is complete and $\gamma_\alpha(u)$ is bounded, $\mathrm{Cl}\,\gamma_\alpha(u)$ is compact. As $\Omega_\alpha(u)$ is closed (Theorem 3.4) and contained in $\mathrm{Cl}\,\gamma_\alpha(u)$, $\Omega_\alpha(u)$ is compact.

(ii) Given any sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ and any sequence $\{z_n\}$, $z_n \in f_\alpha(u, t_n)$, $\gamma_\alpha(u)$ bounded and $\{z_n\} \subset \gamma_\alpha(u)$ imply the existence of a subsequence $\{t_m\} \subset \{t_n\}$ such that $\{z_m\}$ is Cauchy. Hence, by completeness of X, $\{z_m\}$ converges to $z \in \Omega_\alpha(u)$. It follows that $\Omega_\alpha(u)$ is not empty.

(iii) Suppose that $\delta[f_\alpha(u,t), \Omega_\alpha(u)] \nrightarrow 0$ as $t \to \infty$. Then, there exist $\varepsilon > 0$ and a function $g: \mathbb{R}_+ \to X$ such that $g(t) \in f_\alpha(u,t)$, for all $t \in \mathbb{R}_+$, and $d[g(t), \Omega_\alpha(u)] > \varepsilon$. Hence, for all sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, $d[g(t_n), \Omega_\alpha(u)] > \varepsilon$ as $n \to \infty$. Boundedness of $\gamma_\alpha(u)$ implies the existence of a subsequence $\{t_m\}$ of $\{t_n\}$ such that $\{g(t_m)\}$ is Cauchy with $t_m \to \infty$ as $m \to \infty$ and $d[g(t_m), \Omega_\alpha(u)] \to 0$ as $m \to \infty$. This contradicts the supposition. ∎

3.3. Theorems 3.5 and 3.6 provide an effective way of investigating the asymptotic behavior of a fuzzy system. They show that the existence of a real-valued function V satisfying conditions of Theorem 3.5 guarantees the existence of a nonempty compact and α-invariant subset of X, namely, the α-limit set $\Omega_\alpha(u)$, such that

$$\delta[f_\alpha(u,t), \Omega_\alpha(u)] \to 0 \quad \text{as} \quad t \to \infty.$$

Hence, we now desire some method of locating $\Omega_\alpha(u)$. Such a method is provided by the generalization to fuzzy systems of the LaSalle's Invariance Principle [13].

THEOREM 3.7 (Invariance Principle). *Let f be a fuzzy system and let V be a lower-semicontinuous function $X \to \mathbb{R}_+$ such that*

(i) *V is defined in $N \subset X$,*

(ii) *$V(u) > -\infty$, for all $u \in \mathrm{Cl}\,N$,*

(iii) *$\sup D_\alpha V(u) \leqslant -W(u)$, for all $u \in \mathrm{Cl}\,N$, where W is a lower-semicontinuous function $\mathrm{Cl}\,N \to \mathbb{R}_+$.*

If $\Omega_\alpha(u) \subset \mathrm{Cl}\,N$, then $\Omega_\alpha(u) \subset \mathbf{M}$, where $\mathbf{M}$ is the largest α-invariant subset of $\{z \in \mathrm{Cl}\,N: W(z) = 0\}$. If, in addition, $\gamma_\alpha(u)$ is bounded, then $\delta[f_\alpha(u,t), \mathbf{M}] \to 0$ as $t \to \infty$.

Proof. Assuming that $\gamma_\alpha(u) \subset \mathrm{Cl}\,N$, we have $\Omega_\alpha(u) \subset \gamma_\alpha(u) \subset \mathrm{Cl}\,N$. If $\gamma_\alpha(u)$ is not bounded, $\Omega_\alpha(u)$ may be empty, in which case the theorem is obviously true but vacuous. Hence, we will assume that $\Omega_\alpha(u) \neq \varnothing$.

Let $u \in \mathrm{Cl}\, N$. Since $\sup D_\alpha V(u) \leqslant W(u)$, then, for any α-trajectory ϕ_α,

$$\dot{V}(u) = \liminf_{t \to 0_+} \frac{V(\phi_\alpha(u, t)) - V(u)}{t} \leqslant -W(u).$$

Hence, V is nonincreasing along any α-trajectory.

Since V is lower-semicontinuous and ϕ_α is continuous, by a standard result of integration theory, it follows that $V(\phi_\alpha(u, \cdot)): \mathbb{R}_+ \to \mathbb{R}$ is differentiable a.e. on compact subsets of $\mathbb{R}_+$ and that

$$V(\phi_\alpha(u, t)) - V(u) \leqslant \int_0^t \dot{V}(\phi_\alpha(u, \tau))\, d\tau \qquad \text{a.e.} \quad t \in \mathbb{R}_+.$$

Hence, for any sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, for any sequence $\{\phi_\alpha(u, t_n)\}$ such that $\phi_\alpha(u, t_n) \to z \in \Omega_\alpha(u)$ and for a.e. $t \in \mathbb{R}_+$

$$\int_0^t \dot{V}(\phi_\alpha(u, \tau + t_n))\, d\tau \geqslant V(\phi_\alpha(u, t_n + t)) - V(\phi_\alpha(u, t_n)).$$

Defining for all α-trajectory ϕ_α the real

$$\zeta = \inf_{t \in \mathbb{R}_+} V(\phi_\alpha(u, t))$$

yields $V(\phi_\alpha(u, t)) \to \zeta$ as $t \to \infty$, where $V(\cdot) > -\infty$. The lower-semicontinuity of V implies that $\zeta > -\infty$. The uniqueness of the limit ζ shows that

$$\int_0^t \dot{V}(\phi_\alpha(u, \tau + t_n))\, d\tau \to 0 \qquad \text{as} \quad n \to \infty.$$

Let $z \in \Omega_\alpha(u)$ and $\{\phi_\alpha^1(u, t_n)\}$ be the sequence such that $\phi_\alpha^1(u, t_n) \to z$ as $n \to \infty$. Then for all $\phi_\alpha \in \Phi_\alpha$,

$$W(\phi_\alpha(z, \tau)) = W(\phi_\alpha(\lim_{n \to \infty} \phi_\alpha^1(u, t_n)), \tau).$$

As W is lower-semicontinuous, we have

$$W(\phi_\alpha(z, \tau)) \leqslant \liminf_{n \to \infty} W(\phi_\alpha(\phi_\alpha^1(u, t_n), \tau)),$$

that is,

$$W(\phi_\alpha(z, \tau)) \leqslant \liminf_{n \to \infty} W(\phi_\alpha^2(u, t_n + \tau)),$$

where $\phi_\alpha^2(u, t_n + \tau) = \phi_\alpha(\phi_\alpha^1(u, t_n), \tau)$ is an α-trajectory through $\phi_\alpha^1(u, t_n)$.

As W is non-negative and lower-semicontinuous, we have, by applying Fatou's lemma,

$$0 \leqslant \int_0^t W(\phi_\alpha(z, \tau))\, d\tau \leqslant \int_0^t \liminf_{n \to \infty} W(\phi_\alpha^2(u, t_n + \tau))\, d\tau$$

$$\leqslant \liminf_{n \to \infty} \int_0^t W(\phi_\alpha^2(u, t_n + \tau))\, d\tau$$

$$\leqslant - \lim_{n \to \infty} \int_0^t \dot{V}(\phi_\alpha^2(u, t_n + \tau))\, d\tau = 0 \qquad \text{a.e.} \quad t \in \mathbb{R}_+.$$

Therefore, $W(\phi_\alpha(z, \tau)) = 0$ a.e. $\tau \in [0, t]$. The lower-semicontinuity of W implies $W(z) = 0$. Thus, for all $z \in \Omega_\alpha(u)$, $W(z) = 0$, i.e.,

$$\Omega_\alpha(u) \subset \{z \in X \colon W(z) = 0\}. \quad \blacksquare$$

We note that, since $V(\phi_\alpha(u, t)) \to \zeta$ as $t \to \infty$, then, if V is continuous, $V(z) = \zeta$, for all $z \in \Omega_\alpha(u)$. Therefore, if V is continuous, $\Omega_\alpha(u)$ is included in the largest α-invariant subset of $\{z \in X \colon \dot{V}(z) = 0\}$.

4. Stability

4.1. Throughout this section, M denotes a closed subset of X. First recall, for the sake of completeness, the definition of positive-definiteness of a real-valued function [19].

Definition 4.1. A function $V \colon X \to \mathbb{R}$ is said to be *positive-definite* with respect to a closed subset M of X if and only if:

(i) V is defined in a neighborhood N of M.

(ii) $V(u) = 0$, for all $u \in M$.

(iii) For all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that $V(u) < \varepsilon$ whenever $d(u, M) < \delta$.

(iv) There exists a function $\xi \colon \mathbb{R}_+ \to \mathbb{R}$, strictly increasing, continuous and such that $\xi(0) = 0$ and $\xi(d(u, M)) < V(u)$, for all $u \in N - M$.

We note that, since M is closed, $N - M$ is never empty. Hence, there exists $\eta > 0$ such that

$$M \subset B[M, \eta] \subset N.$$

Therefore, if V is a positive-definite function with respect to M and if $K(a)$, $a \in \mathbb{R}_+$, denotes the set

$$K(a) = \{u \in N: V(u) \leqslant a\},$$

it is always possible to find $a = \inf\{V(u): u \in S(M, \eta)\}$ such that $K(a) \subset N$.

DEFINITION 4.2. A subset M of X is said to be *stable* for a fuzzy system f if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that $u \in B(M, \delta)$ implies $f(u, t) \subset B(M, \varepsilon)$, for all $t \in \mathbb{R}_+$.

According to this definition, M is stable if and only if $\gamma(u) \subset B(M, \varepsilon)$ whenever $u \in B(M, \delta)$. However, as seen in the previous sections, $\gamma_\alpha(u)$ may be bounded for $\alpha \geqslant \hat{\alpha}$ and unbounded for $\alpha < \hat{\alpha}$, for some $\hat{\alpha}$. Hence, it may happen that $\gamma_\alpha(u) \subset B(M, \varepsilon)$, for $\alpha \geqslant \hat{\alpha}$ and $\gamma_\alpha(u) \not\subset B(M, \varepsilon)$ for $\alpha < \hat{\alpha}$. Thus, this definition is too large and suggests the removal of the clear distinction between stability and instability, i.e., the introduction of degrees of stability.

DEFINITION 4.3. A subset M of X is said to be *α-stable* for a fuzzy system f if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that $u \in B(M, \delta)$ implies $f_\alpha(u, t) \subset B(M, \varepsilon)$, for all $t \in \mathbb{R}_+$.

DEFINITION 4.4. A subset M of X is an *α-attractor* if there exists a neighborhood N of M such that, for all $u \in N$, for all sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, and all sequence $\{z_n\}$, $z_n \in f_\alpha(u, t_n)$, then $z_n \to z \in M$ as $n \to \infty$.

These two definitions can be rephrased:

(i) A subset M of X is α-stable if and only if, for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that $u \in B(M, \varepsilon)$ implies $\mu_{f(u,t)}(z) \leqslant \alpha$, for all $z \in B(M, \varepsilon)$, for all $t \in \mathbb{R}_+$.

(ii) A subset M of X is an α-attractor if and only if there exists a neighborhood N of M such that, for all $u \in N$, for all sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, and all sequence $\{z_n\}$ such that $\mu_{f(u,t_n)}(z_n) \geqslant \alpha$, then $z_n \to z \in M$ as $n \to \infty$.

DEFINITION 4.5. A subset M of X which is α-stable and is an α-attractor is said to be *α-asymptotically stable.*

We note that if M is (asymptotically) α-stable, then M is (asymptotically) β-stable, for all $\beta > \alpha$.

Criteria of α-stability for a fuzzy system can be established by exploiting our concept of fuzzy derivative of a real-valued function.

THEOREM 4.1. *Let f be a fuzzy system and let M be a subset of X. If there exists a lower-semicontinuous function $V: X \to \mathbb{R}$ such that*

(i) *V is defined in a neighborhood N of M,*

(ii) *V is positive-definite with respect to M,*

(iii) $\sup D_\alpha V(u) \leqslant 0$, *for all $u \in N$,*

then M is α-stable.

Proof. Let $u \in N$. As long as $f_\alpha(u, t) \subset N$, $\sup D_\alpha V(u) \leqslant 0$. Therefore, as long as $f_\alpha(u, t) \subset N$, $V(z) \leqslant V(u)$, for all $z \in f_\alpha(u, t)$.

Let $\hat{\eta} = \inf\{d(z, M): z \in \partial N\}$. Then, for all η such that $0 < \eta < \hat{\eta}$, $K(\eta) \subset \mathrm{Cl}\, N$ and $K(\eta) \cap \partial N = \varnothing$. Let $u \in K(\eta)$. Suppose there exists t such that $f_\alpha(u, t) \not\subset K(\eta)$. Then, there exists τ and $z \in f_\alpha(u, \tau)$ such that $V(z) > \eta$, which is a contradiction. Hence, for all $u \in K(\eta)$, $f_\alpha(u, t) \subset K(\eta)$. Thus, for all $u \in K(\eta)$ and all $z \in f_\alpha(u, t)$, $V(z) \leqslant V(u)$.

V being positive-definite, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $d(u, M) < \delta$ implies $V(u) < \varepsilon$. Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $d(u, M) < \delta$ implies $V(z) < \varepsilon$, for all $z \in f_\alpha(u, t)$.

Moreover, according to Definition 4.1, for all λ there exists $\varepsilon > 0$ such that $d(u, M) > \lambda$ implies $V(u) > \varepsilon = \xi(\lambda)$. Therefore, $V(u) < \varepsilon$ implies $d(u, M) < \lambda$.

Thus, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $d(u, M) < \delta$ implies $d(z, M) < \varepsilon$, for all $z \in f_\alpha(u, t)$, i.e., such that $u \in B(M, \delta)$ implies $f_\alpha(u, t) \subset B(M, \varepsilon)$. ∎

THEOREM 4.2. *Let f be a fuzzy system and let M be a subset of X. If there exists a lower-semicontinuous function $V: X \to \mathbb{R}$ such that*

(i) *V is defined in a neighborhood N of M,*

(ii) *V is positive-definite with respect to M,*

(iii) $\sup D_\alpha V(u) < 0$, *for all $u \in N - M$,*

then M is asymptotically α-stable.

Proof. It suffices to show that M is an α-attractor. To suppose that M is not an α-attractor implies that there exist a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ and a sequence $\{z_n\}$, $z_n \in f_\alpha(u, t_n)$, such that $z_n \to z \in M$. Then, $z \in \Omega_\alpha(u)$. Applying the Invariance Principle shows that $\sup D_\alpha V(z) = 0$, which is a contradiction. ∎

Combining Theorem 4.2 and Theorem 3.7 (Invariance Principle) shows that if M is asymptotically α-stable then, for all $u \in N$, $\Omega_\alpha(u) \subset M$.

Theorems 4.1 and 4.2 suppose the a priori knowledge of the evolution equation and even of the α-trajectories. Since it is rarely the case, it is of prime importance to be able to reformulate these α-stability criteria in such a way that the explicit knowledge of the evolution equation is not required.

THEOREM 4.3. *Let a fuzzy system be defined by a fuzzy relation R and let M be a subset of X. If there exists a C^1-function $V: X \to \mathbb{R}$ such that*

(i) *V is defined in a neighborhood N of M,*

(ii) *V is positive-definite with respect to M,*

(iii) $\sup\{\langle \nabla V, z\rangle : z \in R_\alpha(u)\} \leqslant 0$ *(resp. <0), for all $u \in N - M$,*

then M is α-stable (resp. asymptotically α-stable).

Proof. For all $u \in N - M$, any α-trajectory ϕ_α through u is a solution to the differential relation

$$\dot{\phi}_\alpha(u, t) \in R_\alpha[\phi_\alpha(u, t)] \qquad \text{a.e.} \quad t \in \mathbb{R}_+.$$

Hence, (iii) implies, for all $u \in N - M$,

$$\langle \nabla V, \dot{\phi}_\alpha(u, t)|_{t=0_+}\rangle < 0.$$

Let $\Phi_\alpha^d(u)$ be the set of everywhere differentiable α-trajectories through u. Then, for all $\phi_\alpha \in \Phi_\alpha^d$ and for all $u \in N - M$, there exists a continuous function h such that $h(u) \in R_\alpha(u)$ and $\dot{\phi}_\alpha(u, t) = h(\phi_\alpha(u, t))$. Hence

$$0 > \langle \nabla V, \dot{\phi}_\alpha(u, t)|_{t=0_+}\rangle = \left[\frac{d^+}{dt} V(\phi_\alpha(u, t))\right]_{t=0},$$

and thus, for all $\phi_\alpha \in \Phi_\alpha^d$, $\dot{V}(\phi_\alpha, u) < 0$.

Let ϕ_α be an α-trajectory through u such that $\phi_\alpha \notin \Phi_\alpha^d$, i.e., such that ϕ_α is not differentiable at $t \in \{\tau_1, \tau_2, ...\}$. Then there exist $\phi_\alpha^i \in \Phi_\alpha^d$, $i = 1, 2, ...,$ such that

$$\phi_\alpha(u, t) = \phi_\alpha^i(u, t), \qquad \tau_{i-1} < t < \tau_i,$$

where $\tau_0 = 0$. Hence, for all $\phi_\alpha \in \Phi_\alpha - \Phi_\alpha^d$,

$$\left[\frac{d^+}{dt} V(\phi_\alpha(u, t))\right]_{t=0} < 0.$$

Thus, $\sup D_\alpha V(u) \leqslant 0$, for all $u \in N$. Applying Theorems 4.1 and 4.2 completes the proof. ∎

The α-stability criterion provided by Theorem 4.3 is essentially based upon the relationship which connects ∇V with the derivatives of V along the α-trajectories. However, this result requires the continuous differentiability of the function V. As already mentioned, this assumption is restrictive and then suggests a more general version of this theorem.

THEOREM 4.4. *Let a fuzzy system be defined by a fuzzy relation R and let M be a subset of X. If there exists a lower-semicontinuous function $V: X \to \mathbb{R}$ such that*

(i) *V is defined in a neighborhood N of M,*

(ii) *V is positive-definite with respect to M,*

(iii) *there exists a continuous function $W: X \to \mathbb{R}$ such that*

$$\sup \left\{ \liminf_{\substack{t \to 0_+ \\ \zeta \to z}} \frac{V(u + t\zeta) - V(u)}{t} : z \in R_\alpha(u) \right\} \leqslant -W(u)$$

(resp. $< -W(u)$), for all $u \in N - M$,

then M is α-stable (resp. asymptotically α-stable).

Proof. For all $u \in N$, any α-trajectory ϕ_α through u is a solution to the differential relation

$$\dot{\phi}_\alpha(u, t) \in R_\alpha[\phi_\alpha(u, t)] \qquad \text{a.e.} \quad t \in \mathbb{R}_+.$$

Let Φ_α^d be the set of everywhere differentiable α-trajectories. Then, for all $\phi_\alpha \in \Phi_\alpha^d$ and for all $u \in N$, there exists a continuous function h such that $h(u) \in R_\alpha(u)$ and $\dot{\phi}_\alpha = h(\phi_\alpha)$. Hence, (iii) implies

$$\liminf_{\substack{t \to 0_+ \\ \zeta \to h(u)}} \frac{V(u + t\zeta) - V(u)}{t} \leqslant -W(u),$$

for all $u \in N$.

Hence, applying [29] yields

$$V(\phi_\alpha(u, t)) - V(u) \leqslant -\int_0^t W(\phi_\alpha(u, \tau))\, d\tau$$

and, consequently,

$$\liminf_{t \to 0_+} \frac{V(\phi_\alpha(u, t)) - V(u)}{t} \leqslant -W(u).$$

Thus, for all $\phi_\alpha \in \Phi_\alpha^d$, $\dot{V}(\phi_\alpha, u) \leqslant -W(u)$.

Proceeding as in Theorem 4.3, we can show that, for all $\phi_\alpha \in \Phi_\alpha$, $\dot{V}(\phi_\alpha, u) < -W(u)$. Thus, $\sup D_\alpha V(u) \leqslant -W(u)$. Applying Theorems 4.1 and 4.2 completes the proof. ∎

4.2. The notion of α-stability given by Definition 4.2 implies that for a fuzzy system to be α-stable, all the α-trajectories must have roughly the same behavior (at least in a neighborhood of a subset M of X). The approach developed in Section 3 shows that it may not be the case. This suggests a less restrictive definition of the concept of α-stability which takes into account the possibly different behavior of the α-trajectories.

DEFINITION 4.6. A subset M of X is said to be *partially α-stable* for a fuzzy system f if, for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that $u \in B(M, \delta)$ implies $f_\alpha(u, t) \cap B(M, \varepsilon) \neq \varnothing$, for all $t \in \mathbb{R}_+$.

This definition can be rephrased: a subset M of X is partially α-stable if and only if, for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that if $u \in B(M, \delta)$ then there exists $z \in B(M, \varepsilon)$ such that $\mu_{f(u,t)}(z) > \alpha$, for all $t \in \mathbb{R}_+$.

We note, that unlike α-stability, if M is partially α-stable, then M is partially β-stable for any $\beta < \alpha$.

THEOREM 4.5. *Let f be a fuzzy system and let M be a subset of X. If there exists a lower-semicontinuous function $V: X \to \mathbb{R}$ such that*

(i) *V is defined in a neighborhood N of M,*

(ii) *V is positive-definite with respect to M,*

(iii) $\inf D_\alpha V(u) \leqslant 0$, *for all $u \in N$,*

then M is partially α-stable.

Proof. Since, for all $u \in N$, $\inf D_\alpha V(u) \leqslant 0$, proceeding as in Theorem 4.1, we can show that there exists an α-trajectory $\phi_\alpha \in \Phi_\alpha$ such that $V(\phi_\alpha(u, t)) \leqslant V(u)$, for all $t \in \mathbb{R}_+$.

The positive-definiteness of V implies that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $V(u) < \varepsilon$ whenever $d(u, M) < \delta$. Therefore, for all $\varepsilon > 0$, there exists δ such that $d(u, M) < \delta$ implies $V(\phi_\alpha(u, t)) < \varepsilon$, for all $t \in \mathbb{R}_+$.

Moreover, for all λ, there exists $\varepsilon > 0$ such that (see Definition 4.1) $d(u, M) > \lambda$ implies $V(u) > \varepsilon = \xi(\lambda)$, i.e., such that $d(u, M) < \lambda$ whenever $V(u) < \varepsilon$.

Thus, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $d(u, M) < \delta$ implies $d(\phi_\alpha(u, t), M) < \varepsilon$, for all $t \in \mathbb{R}_+$. Hence, $u \in B(M, \delta)$ implies $\phi_\alpha(u, t) \in B(M, \varepsilon)$, i.e., $f_\alpha(u, t) \cap B(M, \varepsilon) \neq \varnothing$, for all $t \in \mathbb{R}_+$. ∎

Theorem 4.3 provides an α-stability criterion which does not require the explicit knowledge of the evolution equation of the fuzzy system. We can establish a similar result for partial α-stability.

THEOREM 4.6. *Let a fuzzy system be defined by a fuzzy relation R and let M be a subset of X. If there exists a C^1-function $V: X \to \mathbb{R}$ such that*

(i) *V is defined in a neighborhood N of M,*

(ii) *V is positive-definite with respect to M,*

(iii) $\inf\{\langle \nabla V, z\rangle : z \in R_\alpha(u)\} < 0$, *for all $u \in N$,*

then M is partially α-stable.

THEOREM 4.7. *Let a fuzzy system be defined by a fuzzy relation R and let M be a subset of X. If there exists a lower-semicontinuous function $V: X \to \mathbb{R}$ such that*

(i) *V is defined in a neighborhood N of M,*

(ii) *V is positive-definite with respect to M,*

(iii) *there exists a continuous function $W: X \to \mathbb{R}_+$ such that*

$$\inf \left\{ \liminf_{\substack{t \to 0_+ \\ \zeta \to z}} \frac{V(u + t\zeta) - V(u)}{t} : z \in R_\alpha(u) \right\} < -W(u)$$

for all $u \in N$, then M is partially α-stable.

4.3. The concepts of α-stability and of partial α-stability of a subset M of X are of particular interest when M is reduced to an equilibrium point.

DEFINITION 4.7. A point u_e of X is said to be an *α-equilibrium point* of a fuzzy system f if $u_e \in f_\alpha(u_e, t)$ for all $t \in \mathbb{R}_+$.

Clearly, if f is induced by a fuzzy relation R, u_e is an α-equilibrium point if and only if $0 \in R_\alpha(u_e)$.

The following theorems contain several characterizations of an α-equilibrium point.

THEOREM 4.8. *A point u_e of X is an α-equilibrium point of a fuzzy system f if and only if there exists a sequence $\{t_n\}$, $t_n \to 0$ as $n \to \infty$, such that $u_e \in f_\alpha(u_e, t_n)$ for all n.*

Proof. The necessity is obvious. For sufficiency, if $t = kt_n$, for some integer k, then it follows from induction that $f_\alpha(u_e, kt_n) = f_\alpha(f_\alpha(u_e, (k-1)\,t_n), t_n) \ni u_e$. Otherwise, there exists an integer k_n such $k_n t_n < t < (k_n + 1)\,t_n$ and hence an integer m such that $k_n t_n < k_m t_m < t < (k_m + 1)\,t_m < (k_m + 1)\,t_n$. Therefore, $k_n t_n \to t$ as $n \to \infty$. The mapping $f_\alpha(u, \cdot)$ being continuous $d[f_\alpha(u_e, k_n t_n), f_\alpha(u_e, t)] \to 0$ as $n \to \infty$. Thus $u_e \in f_\alpha(u_e, t)$. Since t is arbitrary, u_e is an α-equilibrium point. ∎

THEOREM 4.9. *If u is not an α-equilibrium point of a fuzzy system f, then* *there exist neighborhoods N_1 of u and N_2 of $f_\alpha(u,t)$ such that $N_1 \cap N_2 = \varnothing$ and $f_\alpha(N_1, t) = N_2$.*

Proof. Let t be fixed but arbitrary. Let A and B be two disjoint open subsets of X containing u and $f_\alpha(u,t)$, respectively. Then

$$f_\alpha(A,t) = \bigcup_{\phi_\alpha} \phi_\alpha(A,t).$$

The mapping $z \mapsto \phi_\alpha(z,t)$ being one-to-one and continuous, $\phi_\alpha(A,t)$ is open. Hence, $f_\alpha(A,t)$ is open. Define

$$N_2 = f_\alpha(A,t) \cap B.$$

Then N_2 is an open neighborhood of $f_\alpha(u,t)$ since both $f_\alpha(A,t)$ and B contain $f_\alpha(u,t)$ and are open. Define now

$$N_1 = \{z \in X: \phi_\alpha(z,t) \in N_2 \; \forall \phi_\alpha \in \Phi_\alpha\}.$$

Then

$$N_1 = \bigcap_{\phi_\alpha \in \Phi_\alpha} \{z \in X: \phi_\alpha(z,t) \in N_2\}$$

and hence $f_\alpha(N_1, t) = N_2$. We note that N_1 is not necessarily open but its interior contains u. Since finally A includes N_1, $N_1 \cap N_2 = \varnothing$. ∎

THEOREM 4.10. *The set of α-equilibrium points of a fuzzy system f is closed.*

Proof. If the set of α-equilibrium points is not closed, then there exists a sequence $\{u_n\}$ of α-equilibrium points such that $u_n \to u$ and u is not an α-equilibrium point. Hence, applying Theorem 4.9, there exist two neighborhoods N_1 of u and N_2 of $f_\alpha(u,t)$ such that $N_1 \cap N_2 = \varnothing$ and $f_\alpha(u,t) = N_2$. Since $u_n \to u$, for sufficiently large n, $u_n \in N_1$. Then, we have $f_\alpha(u_n, t) \subset N_2$ and hence $f_\alpha(u_n, t) \cap N_1 = \varnothing$. Thus $u_n \notin f_\alpha(u_n, t)$ which is a contradiction. ∎

THEOREM 4.11. *A point u_e of X is an α-equilibrium point of a fuzzy system f if and only if every neighborhood of u_e contains the orbit of an α-trajectory.*

Proof. The necessity is obvious. For sufficiency, suppose that u_e is not an α-equilibrium point. Then there exist two neighborhoods N_1 of u_e and N_2 of $f_\alpha(u_e, t)$ such that $N_1 \cap N_2 = \varnothing$ and $f_\alpha(N_1, t) = N_2$. Since N_1 contains the orbit of an α-trajectory ϕ_α through u_e, then $\phi_\alpha(z,t) \in N_2$ for all $z \in N_1$. Thus $N_1 \cap N_2 \neq \varnothing$, which is a contradiction. ∎

THEOREM 4.12. *Let f be a fuzzy system. If there exists an α-trajectory* ϕ_α *such that* $d[\varphi_\alpha(u, t), u_e] \to 0$ *as* $t \to \infty$, *then* u_e *is an α-equilibrium point.*

Proof. Let N be a neighborhood of u_e. Since we have $d[\varphi_\alpha(u, t), u_e] \to 0$ as $t \to \infty$, there exists $\mathscr{T}$ such that $\varphi_\alpha(u, t) \subset N$ for all $t \geqslant \mathscr{T}$. Let $z = \phi_\alpha(u, \mathscr{T})$. Then N contains the orbit of $\varphi_\alpha(z, t)$. Applying Theorem 4.11 completes the proof. ∎

Criteria of (partial) α-stability of an α-equilibrium point can be derived in a straightforward manner.

THEOREM 4.13. *Let* u_e *be an α-equilibrium point of a fuzzy system defined by the fuzzy relation R. Let V and W be functions defined as in Theorem* 4.4. *If for all u in a neighborhood of* u_e

$$\sup \left\{ \liminf_{\substack{t \to 0_+ \\ \zeta \to z}} \frac{V(u + t\zeta) - V(u)}{t} : z \in R_\alpha(u) \right\} \leqslant -W(u)$$

(*resp.* $< -W(u)$), *then* u_e *is α-stable* (*resp. asymptotically α-stable*).

THEOREM 4.14. *Let* u_e *be an α-equilibrium point of a fuzzy system defined by a fuzzy relation R. Let V and W be functions defined as in Theorem* 4.7. *If for all u in a neighborhood of* u_e

$$\inf \left\{ \liminf_{\substack{t \to 0_+ \\ \zeta \to z}} \frac{V(u + t\zeta) - V(u)}{t} : z \in R_\alpha(u) \right\} \leqslant -W(u),$$

then u_e *is partially α-stable.*

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